MLC: HOW TO SOLVE IT

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Questions by Topic and Difficulty Level
Fall, 2017

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10 9 8 7 6 5 4 3 2

Second edition: May 2017

Printed in the United States of America

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Preface

Intuition alone is not enough to pass MLC

In the pre-2014 MLC exams, all questions were multiple choice. Developing intuition was paramount and developing rigor was thrown out of the window. You never needed to memorize how to precisely define a symbol or rigorously prove a formula; intuition was all you needed to pass a multiple choice exam. Now tables were turned and written answer questions count for 60% of the MLC exam points. To pass MLC, you need to build intuition and, more importantly, rigor as rigor is far more difficult to build than intuition.

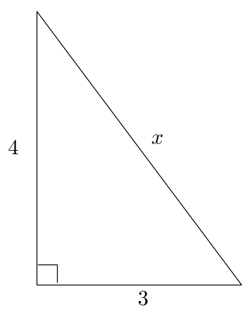
When SOA added written answer questions to MLC in 2014, it was a paradigm shift. Gone were the days when an exam candidate just needed to understand the essence of actuarial concepts without having to worry about how to rigorously define anything. Unfortunately, many students still study for the MLC the old ways. This is how a typical student, Jones, studies for in MLC:

- Jones' focus is on understanding concepts intuitively without any concern on how to define anything rigorously.
- Jones spends most of his study time on perfecting multiple choice questions.
  - It's easy for Jones to find tons of old SOA or CAS multiple choice questions to hone his skill
  - Multiple choices are fun because Jones can use the power of elimination, not to mention there are loads of shortcuts for him to use (such as constant force of mortality shortcuts in $\overline{A}_x$ and $\overline{a}_x$).
- It was only one month before the actual exam when he tried a newly released MLC exam under the exam condition. He scored well in the multiple choice part of the exam, but he did poorly in written answer questions. To his surprise, Jones discovered that written answer questions were the real enemy, but it was too late to turn the tide.
- The final exam day arrived. Though none of the constant force of mortality shortcuts he memorized were tested in the exam, he aced the multiple choice section nonetheless. However, he failed miserably in the written answer part. SOA seemed to know exactly where to poke Jones' weakness.
- Jones failed and now is restudying for MLC.

why written answer questions are hard

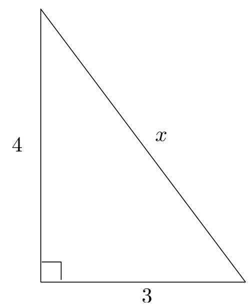
Suppose you need to answer two sets of questions, one set is multiple choice and the other written answer.

Example 0.0.1



What is the value of x?
$(A)3 \quad (B)4 \quad (C)5 \quad (D)6 \quad (E)7$

Example 0.0.2



(a) (3 points) Derive the Pythagorean theorem from the first principle. In addition, show that $x = 5$.

(b) (4 points) Aliens captured the earth and threatened to annihilate the human race unless they were convinced that humans made at least one important discovery. You were sent by your follow earthmen to convince aliens that the Pythagorean theorem was a vital discovery. Explain why Pythagorean theorem is an important discovery.

Example 0.0.3

For a fully discrete whole life insurance of 100,000 on (50), you are given:

(i) The gross premium is calculated under the equivalence principle.

(ii) Expenses, payable at the beginning of the year, are:

	% of Premium	Per Policy
First Year	40%	25
Renewal	5%	10

(iii) Claim cost: 100 per death.

(iv) Mortality: Illustrative Life Table.

(v) $i = 0.06$

Calculate the expense premium for this policy.
$(A)120 \quad (B)170 \quad (C)220 \quad (D)270 \quad (E)320$

Calculate the expense premium policy value at the end of Policy Year 10.
$(A) - 600 \quad (B) - 200 \quad (C)200 \quad (D)600 \quad (E)1,000$

Example 0.0.4

For a fully discrete whole life insurance of 100,000 on (50), you are given:

(i) The gross premium is calculated under the equivalence principle.

(ii) Expenses, payable at the beginning of the year, are:

	% of Premium	Per Policy
First Year	40%	25
Renewal	5%	10

(iii) Claim cost: 100 per death.

(iv) Mortality: Illustrative Life Table.

(v) $i = 0.06$

(a) (2 *points*) Define the expense premium.

(b) (3 *points*) Show that the prospective expense premium policy value at the end of Policy Year 10 is -600 to the nearest of 10. Explain why the expense premium policy value at the end of Policy Year 10 is negative.

(c) (2 *points*) Prove that the retrospective expense premium policy value and the prospective expense premium policy value at the end of Policy Year 10 are equal.

The cognitive skill required to solve a written answer question is significantly higher than what is required to solve a multiple choice question.

intuition vs. rigor

Let's use the concept of the force of mortality and explore the difference between intuitive thinking and rigorous thinking.

What's the force of mortality? Well, it's kind of like the force of interest. The force of interest is an instantaneous rate of increase of your money in a savings account. The force of mortality is an instantaneous rate of increase of what? Yes, an instantaneous rate of increase of the number of survivors ℓ_x or an instantaneous rate of increase of the survival function.

But wait! There's a catch. While money grows over time, the survival function decreases over time – people die over time. So we need to add a negative sign to prevent the force of mortality from becoming negative. No body likes negative numbers. All I need to do is to translate the force of interest formula

$$A(t) = A(0) \exp\left(\int_0^t \delta(s) ds\right)$$

.

into the force of mortality formula:

$$_tp_x = {}_0p_x \exp\left(-\int_0^t \mu_x(s) ds\right)$$

I got it! What else? Wait! The next formula is like a definition. I shall memorize it.

$$\mu_x(t) = -\frac{d}{dt} \ln {}_tp_x$$

That's about it. I definitely don't want to clog my head with useless proofs or fancy jargons invented by scholars who live in the cloud. Next, why don't I solve a bunch of (multiple choice) problems to cement my understanding?

Here's a rigorous approach to the force of mortality:

Let T_0 represent the future lifetime of a newborn. Let T_x represent the future lifetime of life aged x. The force of mortality at age x is represented by μ_x. We define μ_x as:

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} Pr[T_0 \leq x + dx \mid T_0 > x]$$

$$\text{However,} \quad Pr[T_x \leq t] = Pr[T_0 \leq x + t \mid T_0 > x]$$

$$\Rightarrow \mu_x = \lim_{dx \to 0^+} \frac{1}{dx} Pr[T_x \leq dx]$$

Let $S_x = P(T_x > t)$ represent the survival function of T_x. The above equation can be rewritten as:

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \big(1 - S_x(dx)\big)$$

$$\dots$$

While intuitive thinking has served you well in P and FM, it's no longer sufficient for passing MLC. SOA wants to encourage the next generation of actuaries to be articulate, precise, and resourceful in their professions who can handle anything thrown at them. It doesn't want to send a bunch of smart calculators to the ASA and FSA destinations.

don't second guess what SOA will test you

While multiple choice questions tend to be more predictable over time, for written answer questions the sky is the limit. SOA can ask you to do anything, from defining a rudimentary concept such as ${}_tp_x$, to explaining an obscure idea of zeroized reserves, to the dreaded work of deriving Thiele's differential equation. To bring this idea home, let's look at the following example:

Example 0.0.5

(MLC Fall 2015 Q5) (10 points) Dana buys a Type B universal life contract of 100,000. You are given:

(i)

Policy Year k	Annual Premium	Annual Cost of Insurance Rate Per 1000	Percent of Premium Charge	Annual Expense Charge	Surrender Charge
1	1000	——	60%	——	——
2	P_2	2	10%	10	200
3	P_3	3	10%	10	100
$k \geq 4$	P_k	k	5%	10	0

(ii) The credit interest rate is $i^c = 0.06$

(iii) Dana's account value at the end of year 1 is 165.

(iv) Except as indicated, there are no deaths or surrenders.

(a) (2 points) Show that if P_2 were 1000, Dana's account value at the end of year 2 would be 920 to the nearest 10. You should calculate the account value to the nearest 1.

(b) (2 points) Dana's account value at the end of year 3 can be expressed as $aP_2 + bP_3 + c$. Calculate a, b, and c.

(c) (4 points) In year 2, Dana pays a premium of 1000 with probability 0.6, or 200 with probability 0.4.

If he paid 1000 in year 2, then in year 3 he will pay either 1000 with probability 0.6, or 200 with probability 0.4.

If he paid 200 in year 2, then in year 3 he will pay either 1000 with probability 0.2, or 200 with probability 0.8.

(i) Calculate the expected death benefit payable at the end of year 3, if Dana dies then.

(ii) Calculate the expected surrender benefit payable at the end of year 3, if Dana surrenders the contract then.

(d) (2 points) Dana's identical twin, Mark, buys a contract identical to Dana's. If Mark pays 1000 every year, Mark's account value at the end of year 10 will be 5114.

Mark will pay premiums of 1000 in 9 of the first 10 years. Mark will pay no premium in one year, with the year of no premium equally likely to be year 3 or year 10.

Calculate Mark's expected surrender value at the end of year 10.

The setup of this question, especially Part (b), (c), and (d), is simply the work of a genius. You have a better chance of getting struck by lightning than guessing that SOA will test your UL account value this way. To score this problem, you just have to know the UL account value inside out. Now memorized shortcut can save you.

how to prepare for written answer questions

First, from Day One, say goodbye to the pure intuition based learning and embrace rigor in your study. Learn how to define and derive. Build a new habit of showing your work so a random grader can follow your steps.

Next, embrace the pain of studying for MLC. There's no shortcut to learning policy values, profit testing, or any other major concept in MLC. You just have to spend a lot of time learning major concepts and solving problems.

how this book can help you

Imagine that you are to play a chess with a devil. If you win at least 6 out of 10 games, you move on with your life. Otherwise, your life comes to screeching halt and you go to your basement to relearn chess and prepare for another match. How can you beat the devil?

The only thing I can think of is to become a well-rounded chess player. Play a lot of games and experience a lot of fighting scenarios so you can adapt to change.

In this book, I throw you into a wide range of problems, some written answer and others multiple choice style questions. The goal is to help you build both intuition and rigor necessary for passing MLC.

Before writing this book, I went through the SOA syllabus and the AMLCR textbook and identified the major concepts you need to learn to score well in both the written answer questions and the multiple choice questions for the exam. I then wrap these concepts in a series of problems.

In each chapter, I'll teach you how to solve one major type of written answer or multiple choice questions. If you can master the problems in my book, you'll have gained a sophisticated understanding of the core concepts and should be able to tackle most problems SOA throws at you, whether the questions are multiple choice or written answer ones.

My story

I came to U.S. from China when I was in my late twenties. I started my first corporate job in U.S. in the IT department of a large insurance company. After working there for about 2 years, I was ready to switch my career. If you have ever worked in a large IT department or any large department of a large company, you'll find that there are thousands of people just like you who go to the same building in the morning around the same time you go to the building and who leave the same building in the afternoon around the same time you leave the building. The company's giant parking lots were filled with thousands of cars, one of which was my second hand red Toyota Camry. The building is nice. Coworkers are nice. But I felt like a drop of water in the ocean.

I was floating around not sure how to make use of my life. Then one day I heard the actuary profession. I heard that if you were an actuary, you were among the elite group because there weren't enough actuaries to go around. I was interested. I decided to study for P. By that time, I hadn't touched calculus for 13 years. Fortunately, it took me just a couple of months to relearn calculus. I took P and got a 9. I was overjoyed. I applied for a job in the actuary department and became an actuary.

When I became an entry level actuary, I was in my early 30's, about 8 years older than most of my peers, who got the actuary job straight from college. To quickly pass actuary exams, I used a bold strategy: reverse engineering. This is not for the faint of the heart. Think twice before you try it. It works like this. Before I took an exam, say MLC, I used a company printer and printed out all the released MLC exam papers and the official solution papers. There was a stack of paper on my desk. From the stack, I pulled out the most recent exam paper, looked up the SOA solutions, looked up the subject from the textbook, and studied the subject. Then I moved to the next exam paper. I call this just-in-time study, similar to the just-in-time inventory method used in Toyota and many other auto manufacturing plants around the world.

It typically took me two to three months to master all the released papers. When the final exam day came, I walked into the exam room. You know what I saw? Sure there were surprise problems, but most exam problems were just like the problems tested before. I was able to solve those similar problems pretty much 100% right. I just passed another exam.

However, since written answer questions were added in 2014 and there aren't many released written answer questions to master. My old strategy may not work any more.

how to pass MLC or any actuary exam

Based on my experience of studying for actuary exams, I firmly believe that to pass an actuary exam you need to do 2 things: (1) you have to understand the core concepts, and (2) you have to be able to quickly solve the types of problems SOA likes to test.

Building a coherent body of knowledge of the subject matter is the most critical and the most time-consuming part of studying for an actuary exam. If you walk into the exam room muddleheaded or with scanty knowledge of basic theories, none of the tips or tricks you learned from an exam prep book would save you. Any chess master will tell you that there are no shortcuts in learning chess. You just have to know your stuff!

However, knowing the subject well doesn't guarantee passing the exam or earning a high grade any more than good technical skills guarantee a job offer. It's a sad reality that often those who know how to play the interview game get the job. When you take actuary exams, your knowledge is measured by your ability to solve the SOA style questions. To pass MLC, you'll need to immerse yourself in the types of problems SOA likes to test or you'll be one of those "theory smart, exam poor" people.

One key part of studying the SOA exam papers is to identify commonly tested problem types and learn how to solve them quickly. For example, finding the UL account value is tested in virtually every exam. Your first round of effort is to understand what is UL, what is Type A and Type B, what is COI, what is corridor, and how the UL account value builds up over time. After you understand these basic concepts, you face a choice about how to find the Type A UL account value. Should you solve two linear equations on AV_t and COI_t or should you memorize the formulas for AV_t and COI_t to avoid having to solve two equations in the exam? You might try both approaches and see which method suits you. You might find that there's no clear winner and that you want to learn both. However, before taking the exam, you must have a tried-and-true procedure for calculating the Type A UL account value. You don't want to walk into exam empty handed without a proven method in your head.

Here's the final point. It's not absolutely necessary, but it helps. Most people's performance will downgrade in the heat of the exam. To be safe, strive to learn at least a little bit more than the minimal knowledge required to pass MLC. When I was studying an old exam paper, I often asked myself "How can I make this problem harder?" If I saw a subject that was in the syllabus but that was not tested in the past, I often forced myself to learn at least a little bit about it. Even if the subject didn't show up in the test, knowing that I was not a complete idiot on the subject reduced my anxiety.

know your stuff

Google Maps are handy especially when you go to a new place, but I hope you know how to get to your work or school when you left your phone at home or there's no internet connection. Over the years, I have developed many alternative routes for my daily commute. If route A is closed, I know what an alternative route to go. I know which road tends to be jammed by school buses, which road is more likely to have accidents, which alley is slippery when it snows. This knowledge serves me well. Just the other day, while I was driving to work, the road I took had a car accident. While most drivers were stuck in the traffic, I knew the exact small neighborhood that I needed to turn to bypass the traffic jam.

In virtually every career you choose, there's no substitute for learning the basics for doing the job. To study for MLC, you just have to learn the fundamentals: the force of mortality, the multiple state model, profit testing, to name a few.

man's never-ending quest for shortcuts

When I was a college student in China, one day I found a great shortcut for learning English. I'm sure many of you have attempted, at some point in your life, to be very good at a foreign language that is fundamentally different from your native language. A foreign language is like a bottomless pit. No matter how much effort you put into it, you almost always end up going nowhere. You are good enough to say "How are you?" but never be able to understand a movie. You are so angry with yourself and with the foreign language.

Anyway, one day I had a great revelation. If I could spend a year memorizing an English dictionary, I would master the English language once for all! The first few weeks were great. I felt my vocabulary grew exponentially at least for the words that started with the letter A. However, two months later my balloon of hope was punctured. I couldn't move beyond the letter A. And I forgot most of the words I learned. I went back to square one.

There are shortcuts for solving some problems in MLC (such as constant force of mortality shortcuts). However, there's no shortcut for building a coherent body of knowledge for life contingency theories. You just have to learn one concept at a time. You just have to cycle through the major MLC concepts several times to achieve sophisticated understanding.

Most of you reading this book should plan to spend at least 3 months to study for MLC. Learning takes time. That said, if you are a high achiever or you got a 5 last time and are re-taking MLC, 2 months might be enough.

a simple procedure beats the best mind

I remember a story I learned from a computer programming book. The story goes like this. A town in the Midwest has two coffee shops, A and B. If you visit Shop A, sometimes you can get coffee right away but other times they run out of coffee and you have to wait a little while. Shop B, on the other hand, always has coffee ready for a customer who just walks in. Both shops are in the same town and their workers have roughly the same skills. How does Shop B outsmart Shop A? In turns out that Shop B has a simple procedure. If you work in Shop B, from Day One you learn this rule: when existing coffee in a container reaches a certain low level, stop whatever you are doing and immediately start brewing new coffee. This procedure makes all the difference.

A procedure in programming is called an algorithm. When studying for an actuary exam, you'll need to build algorithms for commonly tested problems to avoid having to reinvent the wheel in the heat of the exam. When the big exam day comes, most of the problem types in the exam should be familiar to you and your job is just to recall pre-built algorithms. Don't purposely put yourself on the spot without an algorithm for finding the Type A UL account value. You have only several minutes per exam problem and in the heat of the exam it's really hard to invent a solution to an unseen problem type.

acknowledgement

First, I want to thank two actuaries, Nathan Hardiman and Robin Cunningham, for their generosity. They gave me their Arch manual for the then Course 3 or Exam M practically free. Years ago they wrote a really good study manual called Arch for the then MLC. You might not know that of all the exams for ASA, MLC changes most frequently. For example, if you dig through old Course 3 exam papers, you'll find the famous problem of "Lucky Tom finds coins at the Poisson rate of ... per hour." The Poisson distribution or Poisson process was a hot topic dreaded by many. To your relief, SOA dropped the Poisson process from the syllabus. Anyway, Nathan and Robin have their full time corporate actuarial jobs and couldn't keep up with frequent changes in the exam syllabus. Instead of withdrawing their Arch book from the market and letting their brain child die, they decided to give the Arch manual to another author. Since I wrote many study manuals, they gave me the book.

The Arch manual was a turning point for me technically. After downloading their manuscript from my email, I found out that Arch was written in LaTeX, not in Word. That was the first time I saw LaTeX code. At at time, I was looking for a solution to a long standing problem of Word crashing on me. Arch was a god sent. From Arch, I learned LaTeX and switched from Word to LaTeX for my future books.

In addition, I want to thank the many LaTeX contributors for their wonderful packages. Without LaTeX or many of its special packages, this book isn't possible.

Finally, I think you, dear reader, for reading the thoughts and reasoning I came up with after my actuarial day job. I hope you find this book useful. If you end up using this book, I thank you for the opportunity of being part of your journey into the actuarial dream land.

outlook of actuary profession

According to the U.S. Bureau of Labor Statistics, employment of actuaries is projected to grow 18% from 2014 to 2024, much faster than the average for all occupations. What are you waiting for? Study for MLC today!

FAQ

Does this book cover the entire syllabus?
Yes. The entire syllabus is covered.

Is this book sufficient for passing MLC?
No author can guarantee that if you read his book you will surely pass MLC. That said, if you can master this book and master the SOA exam papers, you have built a solid foundation for passing MLC.

What companion book do you recommend to use along side with this book?
I recommend that you can use this book together with the AMLCR textbook and the SOA exam papers.

errata

Sample chapters and the errata for this book can be found at http://deeperunderstandingfastercalc.com/mlc-solver.php

Chapter 48

Multiple state model: write down Kolmogorov's forward equations 100% right in a hurry

One major pain point that many exam candidates have is to correctly write down the Kolmogorov's forward equations in the heat of the exam. A minor pain point is to write down the associated boundary conditions for the Kolmogorov's forward equations. Today we're going to put an end to these struggles. After reading this chapter, you'll be able to write down the Kolmogorov's forward equations and the boundary conditions 100% correct in a hurry.

Example 48.0.1

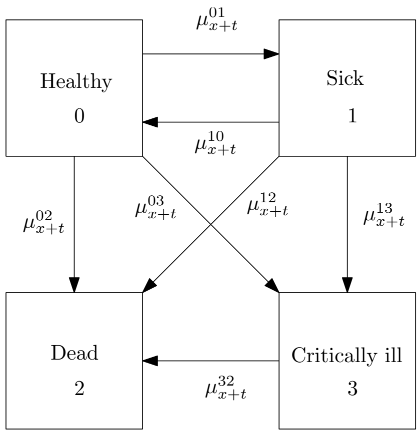
A combined disability and critical illness policy is issued to a healthy life age x. Write down the Kolmogorov's forward equations and the boundary conditions for

(a) ${}_tp_x^{00}$

(b) ${}_tp_x^{01}$

(c) ${}_tp_x^{02}$

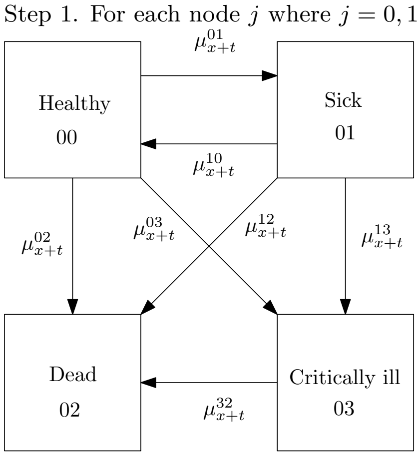
(d) ${}_tp_x^{03}$



Solution 48.0.1

(a) Write down the Kolmogorov's forward equation for ${}_tp_x^{00}$.

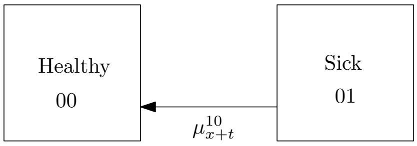
Step 1. For each node j where $j = 0, 1, 2, 3$, change the name of the node from j to $0j$.



We interpret each node as follows: at $t = 0$, the universe has 1 healthy person (ancestor) and no one else. This person's offsprings fill the universe. The population of this universe is always one at any time. Unfortunately, the offspring population isn't growing, unlike Abraham's children. At t, we count the people in the universe.

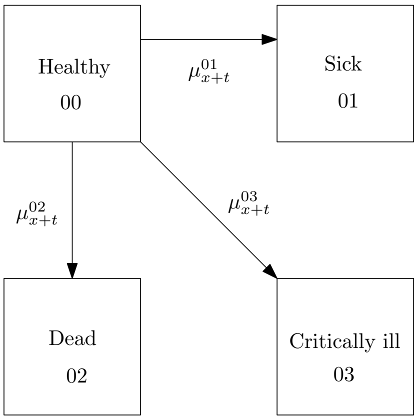
- In the node name $0j$, 0 is the ancestor's name; j is the offspring's name.
- ${}_tp_x^{00}$ is the (expected) number of the healthy people at t.
- ${}_tp_x^{01}$ is the (expected) number of the sick people at t.
- ${}_tp_x^{02}$ is the (expected) number of the dead people at t.
- ${}_tp_x^{03}$ is the (expected) number of the critically ill people at t.
- The total population at t is ${}_tp_x^{00} + {}_tp_x^{01} + {}_tp_x^{02} + {}_tp_x^{03} = 1$.

Step 2. From all the nodes, isolate the nodes that point to the node 00. The insureds in these nodes can flow into the node 00, causing $\frac{d}{dt}{}_tp_x^{00}$ to increase.



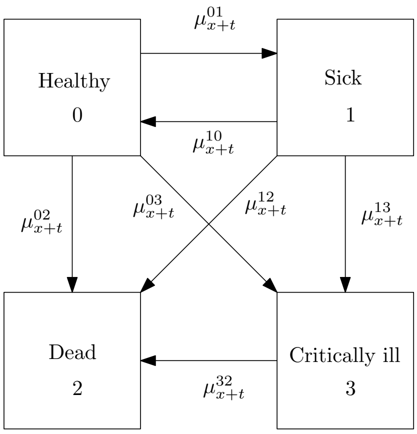
$$\text{positive forces of } \frac{d}{dt}{}_tp_x^{00}: \quad {}_tp_x^{01}\,\mu_{x+t}^{10}$$

Step 3. From all the nodes, isolate the nodes that are pointed to by the node 00. The insureds in the node 00 can flow into these nodes, causing $\frac{d}{dt}{}_tp_x^{00}$ to decrease.



$$\text{negative forces of } \frac{d}{dt}{}_tp_x^{00}: \quad -{}_tp_x^{00}\left(\mu_{x+t}^{01}+\mu_{x+t}^{02}+\mu_{x+t}^{03}\right)$$

Step 4: Combine the positive and the negative forces and you'll get the Kolmogorov's forward equation.

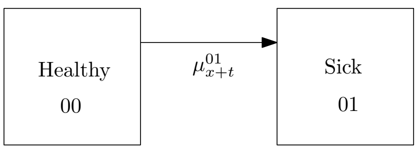


$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\,\mu_{x+t}^{10} - {}_tp_x^{00}\left(\mu_{x+t}^{01}+\mu_{x+t}^{02}+\mu_{x+t}^{03}\right)$$

The boundary condition is either the beginning condition or the ending condition. Since at time zero, the insured is healthy, the boundary conditions are

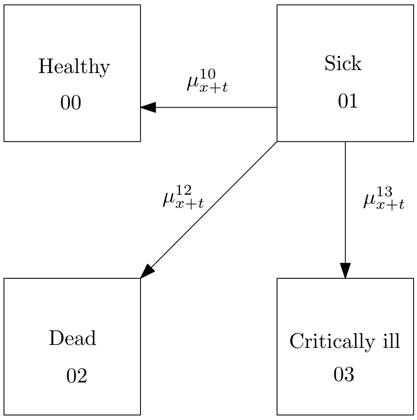
$$ {}_0p_x^{00}=1, \quad {}_0p_x^{01}={}_0p_x^{02}={}_0p_x^{03}=0$$

(b) Write down the Kolmogorov's forward equation for ${}_tp_x^{01}$. Isolate the positive forces (the nodes that can flow into the node 01):



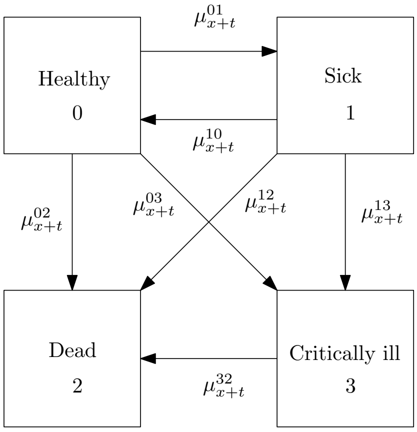
$$\text{positive forces of } \frac{d}{dt}{}_tp_x^{01}: \quad {}_tp_x^{00}\,\mu_{x+t}^{01}$$

Isolate the negative forces (the nodes that the node 01 can flow into):



$$\text{negative forces of } \frac{d}{dt}\,{}_tp_x^{01}: \quad -{}_tp_x^{01}\left(\mu_{x+t}^{10}+\mu_{x+t}^{12}+\mu_{x+t}^{13}\right)$$

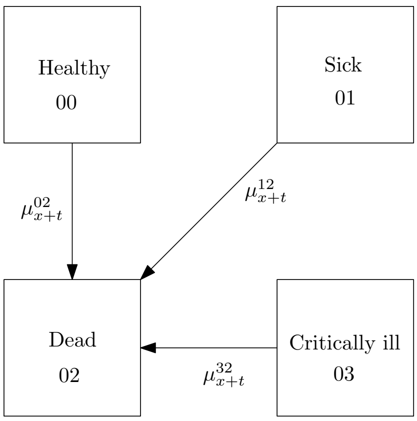
Combine the positive and the negative forces:



$$\frac{d}{dt}\,{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}\left(\mu_{x+t}^{10}+\mu_{x+t}^{12}+\mu_{x+t}^{13}\right)$$

$$\text{boundary condition:} \quad {}_0p_x^{00}=1, \quad {}_0p_x^{01}={}_0p_x^{02}={}_0p_x^{03}=0$$

(c) Write down the Kolmogorov's forward equation for ${}_tp_x^{02}$. Isolate all the nodes that can flow into the node 02:



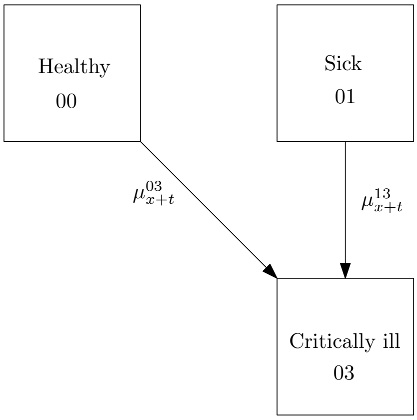
$$\text{positive forces:} \quad \frac{d}{dt}\,{}_tp_x^{02}: \quad {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12} + {}_tp_x^{03}\mu_{x+t}^{32}$$

There are no negative forces that will pull the insured away from the 02 node. The population in the node 02 can only increase.

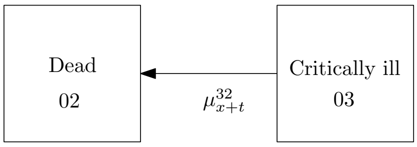
$$\frac{d}{dt}\,{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12} + {}_tp_x^{03}\mu_{x+t}^{32}$$

$$\text{boundary condition:} \quad {}_0p_x^{00}=1, \quad {}_0p_x^{01}={}_0p_x^{02}={}_0p_x^{03}=0$$

(d) Write down the Kolmogorov's forward equation for ${}_tp_x^{03}$.



$$\text{positive forces:} \quad \frac{d}{dt}{}_tp_x^{03}: \quad {}_tp_x^{00}\,\mu_{x+t}^{03} + {}_tp_x^{01}\,\mu_{x+t}^{13}$$



$$\text{negative forces:} \quad \frac{d}{dt}{}_tp_x^{03}: \quad -{}_tp_x^{03}\,\mu_{x+t}^{32}$$

$$\Rightarrow \quad \frac{d}{dt}{}_tp_x^{03} = {}_tp_x^{00}\,\mu_{x+t}^{03} + {}_tp_x^{01}\,\mu_{x+t}^{13} - {}_tp_x^{03}\,\mu_{x+t}^{32}$$

$$\text{boundary condition:} \quad {}_0p_x^{00} = 1, \quad {}_0p_x^{01} = {}_0p_x^{02} = {}_0p_x^{03} = 0$$

Example 48.0.2

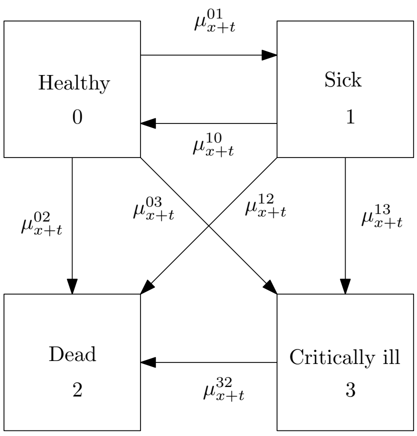
(Same model as before but repeated for convenience) A combined disability and critical illness policy is issued to a healthy life age x. Write down the Kolmogorov's forward equations for

(a) ${}_tp_x^{10}$

(b) ${}_tp_x^{11}$

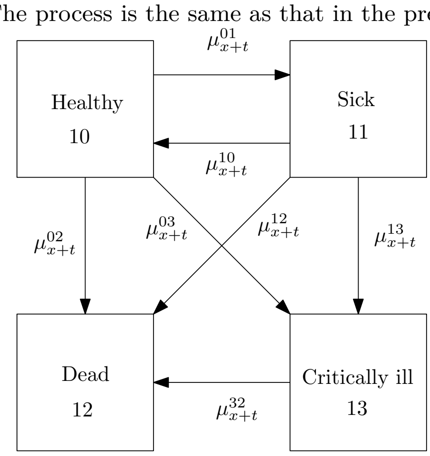
(c) ${}_tp_x^{12}$

(d) ${}_tp_x^{13}$



Solution 48.0.2

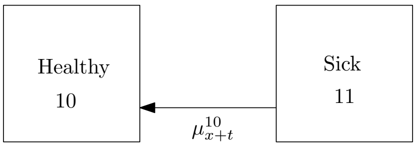
The process is the same as that in the previous problem. First, we relabel each node as $1j$.



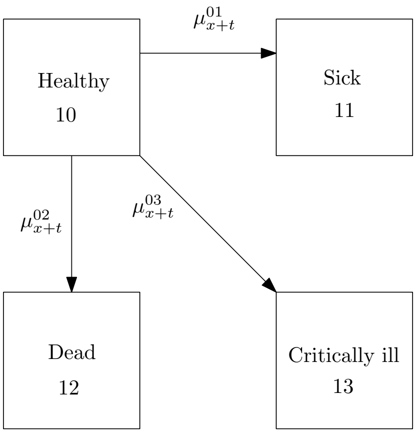
We interpret each node as follows: at some point before t, the universe has only 1 sick person (ancestor). This person's children fill the universe. At t, we count the people in the universe.

- In the node name $1j$, 1 is the ancestor's name; j is the offspring's name.
- ${}_tp_x^{10}$ is the (expected) number of the healthy people at t.
- ${}_tp_x^{11}$ is the (expected) number of the sick people at t.
- ${}_tp_x^{12}$ is the (expected) number of the dead people at t.
- ${}_tp_x^{13}$ is the (expected) number of the critically ill people at t.
- The total population at t is ${}_tp_x^{10} + {}_tp_x^{11} + {}_tp_x^{12} + {}_tp_x^{13} = 1$.

(a)



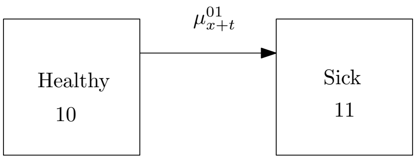
positive forces: $\frac{d}{dt}\,{}_tp_x^{10}$: $\quad {}_tp_x^{11}\,\mu_{x+t}^{10}$



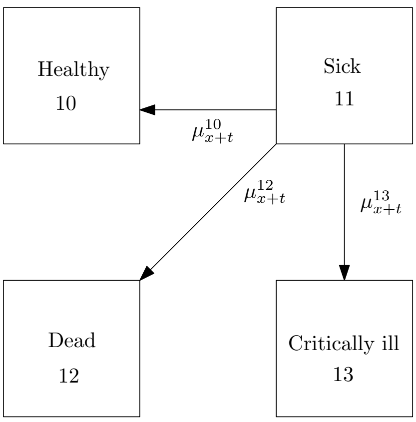
negative forces: $\frac{d}{dt}\,{}_tp_x^{10}$: $\quad -{}_tp_x^{10}\left(\mu_{x+t}^{01}+\mu_{x+t}^{02}+\mu_{x+t}^{03}\right)$

$$\Rightarrow \quad \frac{d}{dt}\,{}_tp_x^{10} = {}_tp_x^{11}\,\mu_{x+t}^{10} - {}_tp_x^{10}\left(\mu_{x+t}^{01}+\mu_{x+t}^{02}+\mu_{x+t}^{03}\right)$$

(b)



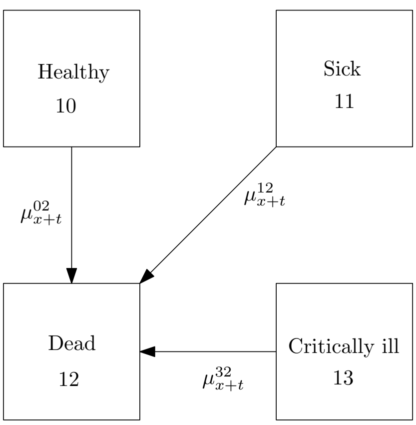
positive forces: $\frac{d}{dt}\,{}_tp_x^{11}$: $\quad {}_tp_x^{10}\,\mu_{x+t}^{01}$



negative forces: $\frac{d}{dt}\,{}_tp_x^{11}$: $\quad -{}_tp_x^{11}\left(\mu_{x+t}^{10}+\mu_{x+t}^{12}+\mu_{x+t}^{13}\right)$

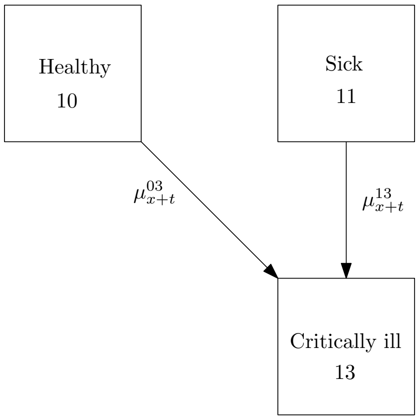
$$\Rightarrow \quad \frac{d}{dt}\,{}_tp_x^{11} = {}_tp_x^{10}\,\mu_{x+t}^{01} - {}_tp_x^{11}\left(\mu_{x+t}^{10}+\mu_{x+t}^{12}+\mu_{x+t}^{13}\right)$$

(c) There are only positive forces for the node 12:

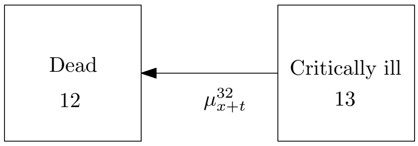


$$\frac{d}{dt}\,{}_tp_x^{12} = {}_tp_x^{10}\,\mu_{x+t}^{02} + {}_tp_x^{11}\,\mu_{x+t}^{12} + {}_tp_x^{13}\,\mu_{x+t}^{32}$$

(d)



positive forces: $\frac{d}{dt}\,{}_tp_x^{13}$: $\quad {}_tp_x^{10}\,\mu_{x+t}^{03} + {}_tp_x^{11}\,\mu_{x+t}^{13}$



negative forces: $\frac{d}{dt}\,{}_tp_x^{13}$: $\quad -{}_tp_x^{13}\,\mu_{x+t}^{32}$

$$\Rightarrow \quad \frac{d}{dt}\,{}_tp_x^{13} = {}_tp_x^{10}\,\mu_{x+t}^{03} + {}_tp_x^{11}\,\mu_{x+t}^{13} - {}_tp_x^{13}\,\mu_{x+t}^{32}$$

Example 48.0.3

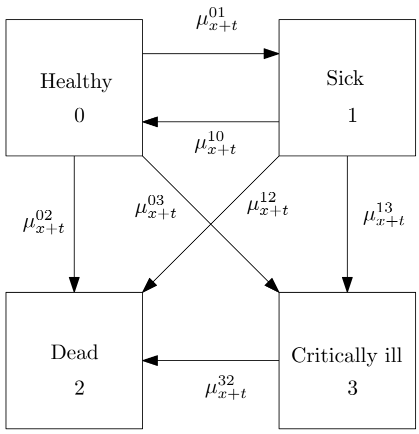
(Same model as before but repeated for convenience) A combined disability and critical illness policy is issued to a healthy life age x. Write down the Kolmogorov's forward equation for

(a) ${}_tp_x^{30}$

(b) ${}_tp_x^{31}$

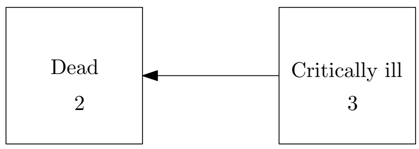
(c) ${}_tp_x^{32}$

(d) ${}_tp_x^{33}$



Solution 48.0.3

If the insured is in the state 3, he can't go to the state 0 or 1. Hence the model can be simplified to:



$${}_tp_x^{30} = {}_tp_x^{31} = 0, \quad \frac{d}{dt}\,{}_tp_x^{30} = \frac{d}{dt}\,{}_tp_x^{31} = 0$$

$$\frac{d}{dt}\,{}_tp_x^{32} = {}_tp_x^{33}\,\mu_{x+t}^{32}$$

$$\frac{d}{dt}\,{}_tp_x^{33} = -{}_tp_x^{33}\,\mu_{x+t}^{32}$$

Example 48.0.4

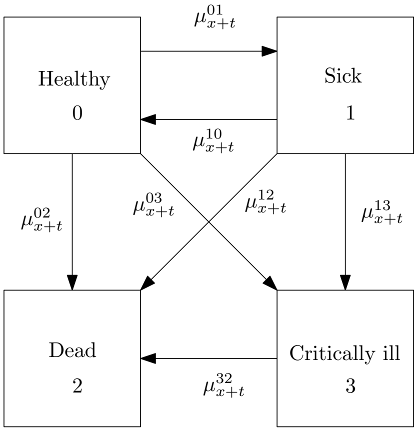
(Same model as before but repeated for convenience) A combined disability and critical illness policy is issued to a healthy life age x. Write down the Kolmogorov's forward equations for

(a) ${}_tp_x^{20}$

(b) ${}_tp_x^{21}$

(c) ${}_tp_x^{22}$

(d) ${}_tp_x^{23}$



Solution 48.0.4

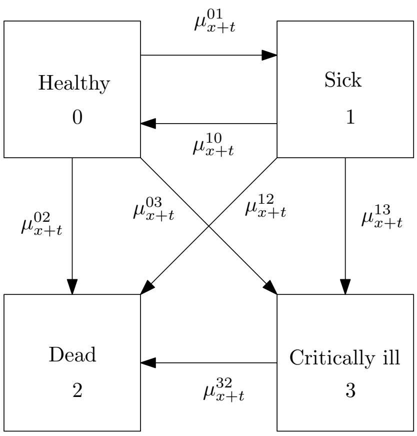
If the insured is currently in the state 2, he'll be stuck in the state 2 and can't go to the state 0, 1 or 3. The model can be simplified to:

$$
\begin{aligned}
& {}_tp_x^{20} = {}_tp_x^{21} = {}_tp_x^{23} = 0, \quad {}_tp_x^{22} = 1 \\
\Rightarrow \quad & \frac{d}{dt}{}_tp_x^{20} = \frac{d}{dt}{}_tp_x^{21} = \frac{d}{dt}{}_tp_x^{22} = \frac{d}{dt}{}_tp_x^{23} = 0
\end{aligned}
$$

Example 48.0.5

(Same model as before but repeated for convenience) A combined disability and critical illness policy is issued to a healthy life age x. The insured is currently in the state 0. Explain why

$$\frac{d}{dt}{}_tp_x^{00} + \frac{d}{dt}{}_tp_x^{01} + \frac{d}{dt}{}_tp_x^{02} + \frac{d}{dt}{}_tp_x^{03} = 0$$



Solution 48.0.5

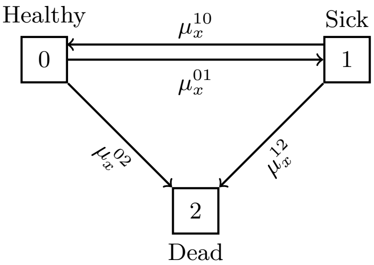
Since the total population is always one, the total change of the population is always zero. Similarly,

$$
\begin{aligned}
\frac{d}{dt}{}_tp_x^{10} + \frac{d}{dt}{}_tp_x^{11} + \frac{d}{dt}{}_tp_x^{12} + \frac{d}{dt}{}_tp_x^{13} = 0 \\
\frac{d}{dt}{}_tp_x^{20} + \frac{d}{dt}{}_tp_x^{21} + \frac{d}{dt}{}_tp_x^{22} + \frac{d}{dt}{}_tp_x^{23} = 0 \\
\frac{d}{dt}{}_tp_x^{30} + \frac{d}{dt}{}_tp_x^{31} + \frac{d}{dt}{}_tp_x^{32} + \frac{d}{dt}{}_tp_x^{33} = 0
\end{aligned}
$$

48.1 Check your knowledge

Homework 48.1.1

Write down the formula for $\frac{d}{dt}{}_tp_x^{01}$.



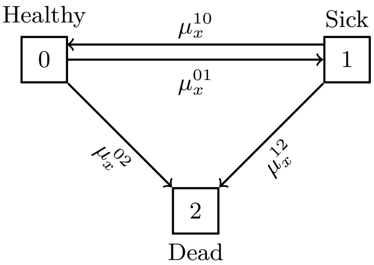
Homework Solution 48.1.1

★★★☆☆ Difficulty

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12})$$

Homework 48.1.2

Write down the formula for $\frac{d}{dt}{}_tp_x^{00}$.



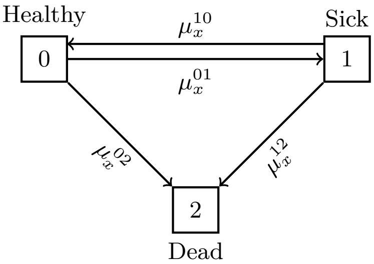
Homework Solution 48.1.2

★★★☆☆ Difficulty

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$$

Homework 48.1.3

Write down the formula for $\frac{d}{dt}{}_tp_x^{22}$.



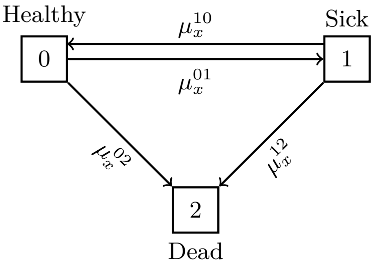
Homework Solution 48.1.3

★★★☆☆ Difficulty

$$\frac{d}{dt}{}_tp_x^{22} = 0$$

Homework 48.1.4

Write down the formula for $\frac{d}{dt}{}_tp_x^{10}$, $\frac{d}{dt}{}_tp_x^{12}$, $\frac{d}{dt}{}_tp_x^{02}$.



Homework Solution 48.1.4

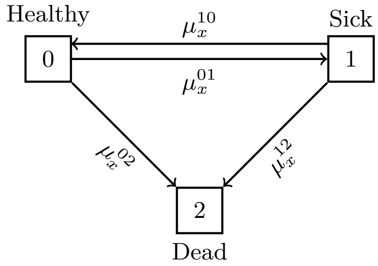
★★★☆☆ Difficulty

$$\frac{d}{dt}{}_tp_x^{10} = {}_tp_x^{11}\mu_{x+t}^{10} - {}_tp_x^{10}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$$
$$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12} + {}_tp_x^{10}\mu_{x+t}^{02}$$
$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12}$$

Homework 48.1.5

Use the Kolmogorov's forward equation to derive the formula:

$$ {}_tp_x^{\overline{00}} = \exp\left(-\int_0^t (\mu_{x+s}^{01} + \mu_{x+s}^{02}) ds \right) $$



Homework Solution 48.1.5

★★★★☆ Difficulty

$$ \frac{d}{dt} {}_tp_x^{00} = {}_tp_x^{01} \mu_{x+t}^{10} - {}_tp_x^{00} (\mu_{x+t}^{01} + \mu_{x+t}^{02}) $$

Since we want the insured to be continuously in the state 0, set ${}_tp_x^{01} = 0$:

$$ \Rightarrow \quad \frac{d}{dt} {}_tp_x^{\overline{00}} = -{}_tp_x^{\overline{00}} (\mu_{x+t}^{01} + \mu_{x+t}^{02}) $$

$$ \frac{1}{{}_tp_x^{\overline{00}}} \frac{d}{dt} {}_tp_x^{\overline{00}} = \frac{d}{dt} \ln {}_tp_x^{\overline{00}} = -(\mu_{x+t}^{01} + \mu_{x+t}^{02}) $$

Integrate both sides:

$$ {}_tp_x^{\overline{00}} = C \exp\left(-\int_0^t (\mu_{x+s}^{01} + \mu_{x+s}^{02}) ds \right) $$

where C is a constant.

$$ {}_0p_x^{\overline{00}} = 1, \quad \Rightarrow C = 1 $$

$$ {}_tp_x^{\overline{00}} = \exp\left(-\int_0^t (\mu_{x+s}^{01} + \mu_{x+s}^{02}) ds \right) $$

Chapter 49

Multiple state model: Euler's method for probabilities

Example 49.0.1

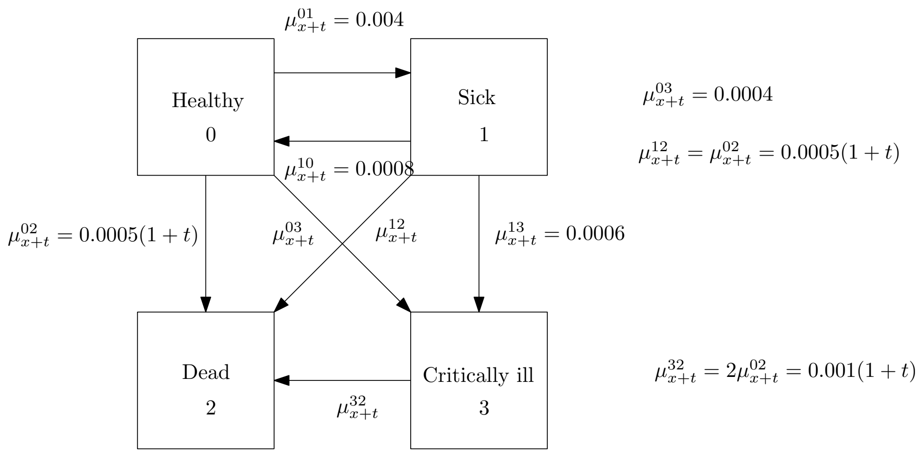
A combined disability and critical illness policy is issued to a healthy life age x. Let $h = 1/12$ (e.g. one month time step). Use the Euler's method and estimate the following probabilities:

(a) ${}_hp_x^{00}$

(b) ${}_hp_x^{01}$

(c) ${}_hp_x^{02}$

(d) ${}_hp_x^{03}$



Solution 49.0.1

The boundary conditions are:

$${}_0p_x^{00} = 1, \quad {}_0p_x^{01} = {}_0p_x^{02} = {}_0p_x^{03} = 0$$

(*a*)

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03}\right)$$

$$\Rightarrow \quad \left(\frac{d}{dt}{}_tp_x^{00}\right)_{t=0} = {}_0p_x^{01}\mu_{x+0}^{10} - {}_0p_x^{00}\left(\mu_{x+0}^{01} + \mu_{x+0}^{02} + \mu_{x+0}^{03}\right)$$

$$= 0\mu_{x+0}^{10} - 1\left(\mu_{x+0}^{01} + \mu_{x+0}^{02} + \mu_{x+0}^{03}\right)$$

$$= 0 - 1\left(0.004 + 0.0005(1+0) + 0.0004\right) = -0.0049$$

$${}_hp_x^{00} \approx {}_0p_x^{00} + h\left(\frac{d}{dt}{}_tp_x^{00}\right)_{t=0} = 1 + \frac{1}{12} \times (-0.0049) = 0.99959167$$

(*b*)

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13}\right)$$

$$\left(\frac{d}{dt}{}_tp_x^{01}\right)_{t=0} = {}_0p_x^{00}\mu_{x+0}^{01} - {}_0p_x^{01}\left(\mu_{x+0}^{10} + \mu_{x+0}^{12} + \mu_{x+0}^{13}\right)$$

$$= \mu_{x+0}^{01} = 0.004$$

$${}_hp_x^{01} \approx {}_0p_x^{01} + h\left(\frac{d}{dt}{}_tp_x^{01}\right)_{t=0} = 0 + \frac{1}{12} \times 0.004 = 0.00033333$$

(*c*)

$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12} + {}_tp_x^{03}\mu_{x+t}^{32}$$

$$\left(\frac{d}{dt}{}_tp_x^{02}\right)_{t=0} = {}_0p_x^{00}\mu_{x+0}^{02} + {}_0p_x^{01}\mu_{x+0}^{12} + {}_0p_x^{03}\mu_{x+0}^{32}$$

$$= \mu_{x+0}^{02} = 0.0005(1+0) = 0.0005$$

$${}_hp_x^{03} \approx {}_0p_x^{03} + h\left(\frac{d}{dt}{}_tp_x^{03}\right)_{t=0} = 0 + \frac{1}{12} \times 0.0005 = 0.00004167$$

(d)

$$\frac{d}{dt}{}_tp_x^{03} = {}_tp_x^{00}\mu_{x+t}^{03} + {}_tp_x^{01}\mu_{x+t}^{13} - {}_tp_x^{03}\mu_{x+t}^{32}$$

$$\left(\frac{d}{dt}{}_tp_x^{03}\right)_{t=0} = {}_0p_x^{00}\mu_{x+0}^{03} + {}_0p_x^{01}\mu_{x+0}^{13} - {}_0p_x^{03}\mu_{x+0}^{32}$$

$$= \mu_{x+0}^{03} = 0.0004$$

$${}_hp_x^{03} \approx {}_0p_x^{03} + h\left(\frac{d}{dt}{}_tp_x^{03}\right)_{t=0} = 0 + \frac{1}{12} \times 0.0004 = 0.00003333$$

$$\text{Check:} \quad 0.99959167 + 0.00033333 + 0.00004167 + 0.00003333 = 1 \quad \text{OK}$$

Example 49.0.2

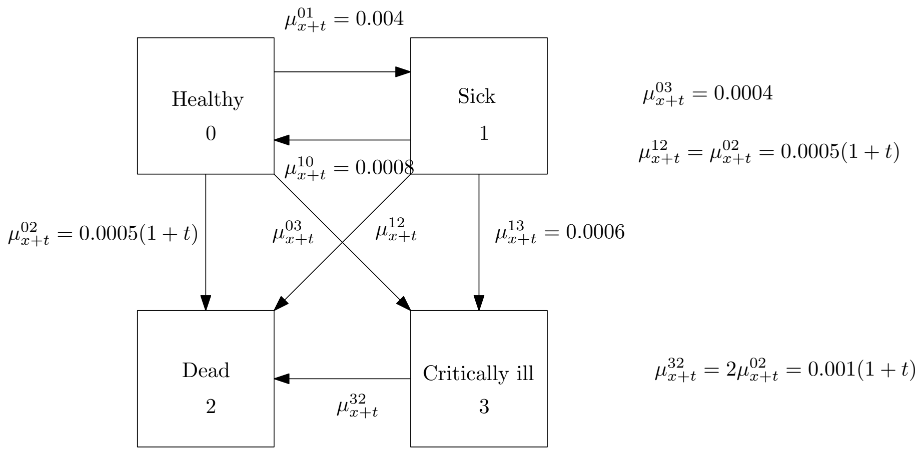
(Same model as before but repeated for convenience) A combined disability and critical illness policy is issued to a healthy life age x. Let $h = 1/12$ (e.g. one month time step). Use the Euler's method and estimate the following probabilities:

(a) ${}_{2h}p_x^{00}$

(b) ${}_{2h}p_x^{01}$

(c) ${}_{2h}p_x^{02}$

(d) ${}_{2h}p_x^{03}$



You are given: ${}_hp_x^{00} = 0.99959167$, ${}_hp_x^{01} = 0.00033333$, ${}_hp_x^{02} = 0.00004167$, and ${}_hp_x^{03} = 0.00003333$

Solution 49.0.2

(a)

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03}\right)$$

$$\Rightarrow \quad \left(\frac{d}{dt}{}_tp_x^{00}\right)_{t=h} = {}_hp_x^{01}\mu_{x+h}^{10} - {}_hp_x^{00}\left(\mu_{x+h}^{01} + \mu_{x+h}^{02} + \mu_{x+h}^{03}\right)$$

$$= 0.00033333 \times 0.0008 - 0.99959167\left(0.004 + 0.0005(1 + 1/12) + 0.0004\right)$$

$$= -0.00493938$$

$${}_{2h}p_x^{00} \approx {}_hp_x^{00} + h\left(\frac{d}{dt}{}_tp_x^{00}\right)_{t=h} = 0.99959167 + \frac{1}{12} \times (-0.00493938) = 0.99918006$$

(b)

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13}\right)$$

$$\left(\frac{d}{dt}{}_tp_x^{01}\right)_{t=h} = {}_hp_x^{00}\mu_{x+h}^{01} - {}_hp_x^{01}\left(\mu_{x+h}^{10} + \mu_{x+h}^{12} + \mu_{x+h}^{13}\right)$$

$$= 0.99959167 \times 0.004 - 0.00033333\left(0.0008 + 0.0005(1 + 1/12) + 0.0006\right)$$

$$= 0.00399772$$

$${}_{2h}p_x^{01} \approx {}_hp_x^{01} + h\left(\frac{d}{dt}{}_tp_x^{01}\right)_{t=h} = 0.00033333 + \frac{1}{12} \times (0.00399772) = 0.000666473$$

(c)

$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12} + {}_tp_x^{03}\mu_{x+t}^{32}$$

$$\left(\frac{d}{dt}{}_tp_x^{02}\right)_{t=h} = {}_hp_x^{00}\mu_{x+h}^{02} + {}_hp_x^{01}\mu_{x+h}^{12} + {}_hp_x^{03}\mu_{x+h}^{32}$$

$$= 0.99959167 \times 0.0005(1 + 1/12) + 0.00033333 \times 0.0005(1 + 1/12) + 0.00003333 \times 0.001(1 + 1/12)$$

$$= 0.00054166$$

$${}_{2h}p_x^{02} \approx {}_hp_x^{02} + h\left(\frac{d}{dt}{}_tp_x^{02}\right)_{t=h} = 0.00004167 + \frac{1}{12} \times 0.00054166 = 0.00008700$$

(d)

$$\frac{d}{dt}{}_tp_x^{03} = {}_tp_x^{00}\mu_{x+t}^{03} + {}_tp_x^{01}\mu_{x+t}^{13} - {}_tp_x^{03}\mu_{x+t}^{32}$$

$$\left(\frac{d}{dt}{}_tp_x^{03}\right)_{t=h} = {}_hp_x^{00}\mu_{x+h}^{03} + {}_hp_x^{01}\mu_{x+h}^{13} - {}_hp_x^{03}\mu_{x+h}^{32}$$

$$= 0.99959167 \times 0.0004 + 0.00033333 \times 0.0006 - 0.00003333 \times 0.001(1 + 1/12)$$

$$= 0.00040000$$

$${}_{2h}p_x^{03} \approx {}_hp_x^{03} + h\left(\frac{d}{dt}{}_tp_x^{03}\right)_{t=h} = 0.00003333 + \frac{1}{12} \times 0.00040000 = 0.00006663$$

$$\text{Check:} \quad 0.99918006 + 0.000666473 + 0.00008700 + 0.00006663 = 1.0000002 \approx 1 \quad \text{OK}$$

Example 49.0.3

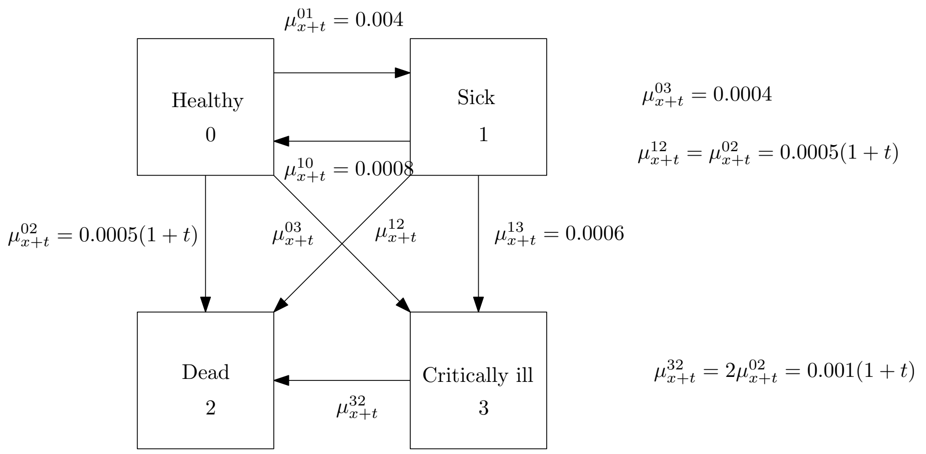
(Same model as before but repeated for convenience) A combined disability and critical illness policy is issued to a healthy life age x. Let $h = 1/12$ (e.g. one month time step). Write down the formulas for each of the following probabilities under the Euler's method. Numerical calculations are not expected.

(a) ${}_{nh}p_x^{00}$

(b) ${}_{nh}p_x^{01}$

(c) ${}_{nh}p_x^{02}$

(d) ${}_{nh}p_x^{03}$



Solution 49.0.3

(a)

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\,\mu_{x+t}^{10} - {}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03}\right)$$

$$\Rightarrow \quad \left(\frac{d}{dt}{}_tp_x^{00}\right)_{t=(n-1)h} = {}_{(n-1)h}p_x^{01}\,\mu_{x+(n-1)h}^{10} - {}_{(n-1)h}p_x^{00}\left(\mu_{x+(n-1)h}^{01} + \mu_{x+(n-1)h}^{02} + \mu_{x+(n-1)h}^{03}\right)$$

$${}_{nh}p_x^{00} \approx {}_{(n-1)h}p_x^{00} + h\left(\frac{d}{dt}{}_tp_x^{00}\right)_{t=(n-1)h}$$

(b)

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\,\mu_{x+t}^{01} - {}_tp_x^{01}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13}\right)$$

$$\left(\frac{d}{dt}{}_tp_x^{01}\right)_{t=(n-1)h} = {}_{(n-1)h}p_x^{00}\,\mu_{x+(n-1)h}^{01} - {}_{(n-1)h}p_x^{01}\left(\mu_{x+(n-1)h}^{10} + \mu_{x+(n-1)h}^{12} + \mu_{x+(n-1)h}^{13}\right)$$

$${}_{nh}p_x^{01} \approx {}_{(n-1)h}p_x^{01} + h\left(\frac{d}{dt}{}_tp_x^{01}\right)_{t=(n-1)h}$$

(c)

$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\,\mu_{x+t}^{02} + {}_tp_x^{01}\,\mu_{x+t}^{12} + {}_tp_x^{03}\,\mu_{x+t}^{32}$$

$$\left(\frac{d}{dt}{}_tp_x^{02}\right)_{t=(n-1)h} = {}_{(n-1)h}p_x^{00}\,\mu_{x+(n-1)h}^{02} + {}_{(n-1)h}p_x^{01}\,\mu_{x+(n-1)h}^{12} + {}_{(n-1)h}p_x^{03}\,\mu_{x+(n-1)h}^{32}$$

$${}_{nh}p_x^{02} \approx {}_{(n-1)h}p_x^{02} + h\left(\frac{d}{dt}{}_tp_x^{02}\right)_{t=(n-1)h}$$

(d)

$$\frac{d}{dt}{}_tp_x^{03} = {}_tp_x^{00}\,\mu_{x+t}^{03} + {}_tp_x^{01}\,\mu_{x+t}^{13} - {}_tp_x^{03}\,\mu_{x+t}^{32}$$

$$\left(\frac{d}{dt}{}_tp_x^{03}\right)_{t=(n-1)h} = {}_{(n-1)h}p_x^{00}\,\mu_{x+(n-1)h}^{03} + {}_{(n-1)h}p_x^{01}\,\mu_{x+(n-1)h}^{13} - {}_{(n-1)h}p_x^{03}\,\mu_{x+(n-1)h}^{32}$$

$${}_{nh}p_x^{03} \approx {}_{(n-1)h}p_x^{03} + h\left(\frac{d}{dt}{}_tp_x^{03}\right)_{t=(n-1)h}$$

Example 49.0.4

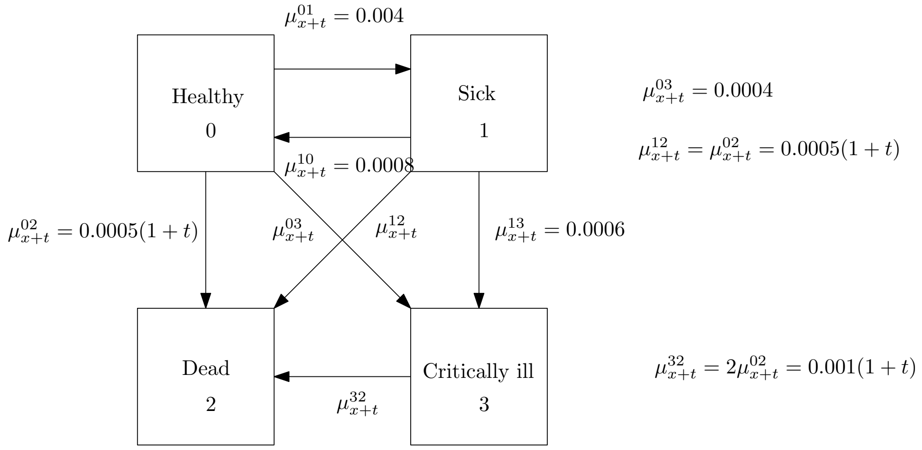
(Same model as before but repeated for convenience) A combined disability and critical illness policy is issued to a healthy life age x. Let $h = 1/12$ (e.g. one month time step). Use the Euler's method and estimate the following probabilities:

(a) ${}_hp_x^{10}$

(b) ${}_hp_x^{11}$

(c) ${}_hp_x^{12}$

(d) ${}_hp_x^{13}$



Solution 49.0.4

It may strike you as absurd to calculate these probabilities given that at time zero (e.g. contract initiation) the insured is in the state 0. However, we can still compile these probabilities by arbitrarily assuming that the insured is in the state 1 at time zero. The boundary conditions for these probabilities are:

$$ {}_0p_x^{11} = 1, \quad {}_0p_x^{10} = {}_0p_x^{12} = {}_0p_x^{13} = 0 $$

(a)

$$ \frac{d}{dt}{}_tp_x^{10} = {}_tp_x^{11}\,\mu_{x+t}^{10} - {}_tp_x^{10}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03}\right) $$

$$ \Rightarrow \quad \left(\frac{d}{dt}{}_tp_x^{10}\right)_{t=0} = {}_0p_x^{11}\,\mu_{x+0}^{10} - {}_0p_x^{10}\left(\mu_{x+0}^{01} + \mu_{x+0}^{02} + \mu_{x+0}^{03}\right) $$

$$ = 1 \times \mu_{x+0}^{10} - 0\left(\mu_{x+0}^{01} + \mu_{x+0}^{02} + \mu_{x+0}^{03}\right) = \mu_{x+0}^{10} = 0.0008 $$

$$ {}_hp_x^{01} \approx {}_0p_x^{01} + h\left(\frac{d}{dt}{}_tp_x^{01}\right)_{t=0} = 0 + \frac{1}{12} \times 0.0008 = 0.00006667 $$

(b)

$$ \frac{d}{dt}{}_tp_x^{11} = {}_tp_x^{10}\,\mu_{x+t}^{01} - {}_tp_x^{11}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13}\right) $$

$$ \left(\frac{d}{dt}{}_tp_x^{11}\right)_{t=0} = {}_0p_x^{10}\,\mu_{x+0}^{01} - {}_0p_x^{11}\left(\mu_{x+0}^{10} + \mu_{x+0}^{12} + \mu_{x+0}^{13}\right) $$

$$ = -\left(\mu_{x+0}^{10} + \mu_{x+0}^{12} + \mu_{x+0}^{13}\right) $$

$$ = -\Big(0.0008 + 0.0005(1+0) + 0.0006\Big) = -0.00190000 $$

$$ {}_hp_x^{11} \approx {}_0p_x^{11} + h\left(\frac{d}{dt}{}_tp_x^{11}\right)_{t=0} = 1 + \frac{1}{12} \times (-0.00190000) = 0.99984167 $$

(c)

$$ \frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{10}\,\mu_{x+t}^{02} + {}_tp_x^{11}\,\mu_{x+t}^{12} + {}_tp_x^{13}\,\mu_{x+t}^{32} $$

$$ \left(\frac{d}{dt}{}_tp_x^{12}\right)_{t=0} = {}_0p_x^{10}\,\mu_{x+0}^{02} + {}_0p_x^{11}\,\mu_{x+0}^{12} + {}_0p_x^{13}\,\mu_{x+0}^{32} $$

$$ = \mu_{x+0}^{12} = 0.0005(1+0) = 0.0005 $$

$$ {}_hp_x^{12} \approx {}_0p_x^{12} + h\left(\frac{d}{dt}{}_tp_x^{12}\right)_{t=0} = 0 + \frac{1}{12} \times 0.0005 = 0.00004167 $$

(d)

$$ \frac{d}{dt}{}_tp_x^{13} = {}_tp_x^{10}\,\mu_{x+t}^{03} + {}_tp_x^{11}\,\mu_{x+t}^{13} - {}_tp_x^{13}\,\mu_{x+t}^{32} $$

$$ \left(\frac{d}{dt}{}_tp_x^{13}\right)_{t=0} = {}_0p_x^{10}\,\mu_{x+0}^{03} + {}_0p_x^{11}\,\mu_{x+0}^{13} - {}_0p_x^{13}\,\mu_{x+0}^{32} $$

$$ = \mu_{x+0}^{13} = 0.0006 $$

$$ {}_hp_x^{13} \approx {}_0p_x^{13} + h\left(\frac{d}{dt}{}_tp_x^{13}\right)_{t=0} = 0 + \frac{1}{12} \times 0.0006 = 0.00005 $$

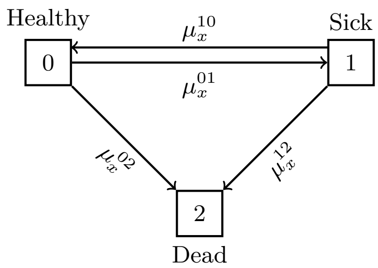
$$ \text{Check:} \quad 0.00006667 + 0.99984167 + 0.00004167 + 0.00005 = 1 \quad \text{OK} $$

49.1 Check your knowledge

Homework 49.1.1

You are given the following model (AMLCR textbook Example 8.5):

- $\mu_x^{01} = a_1 + b_1 \exp(c_1 x)$, $\mu_x^{10} = 0.1\mu_x^{01}$, $\mu_x^{02} = a_2 + b_2 \exp(c_2 x)$, $\mu_x^{12} = \mu_x^{02}$
- $a_1 = 0.0004$, $b_1 = 3.4674 \times 10^{-6}$, $c_1 = 0.138155$
- $a_2 = 0.0005$, $b_2 = 7.5858 \times 10^{-5}$, $c_2 = 0.087498$



Let $h = 1/12$ and $x = 60$.

(a) Briefly explain why it's difficult or impossible to find an exact formula for ${}_hp_x^{00}$.

(b) Briefly explain why the Euler's method works.

(c) Calculate ${}_hp_x^{00}$

(d) Calculate ${}_hp_x^{01}$

(e) Calculate ${}_hp_x^{02}$

(f) Calculate ${}_{2h}p_x^{00}$

(g) Calculate ${}_{2h}p_x^{01}$

(h) Calculate ${}_{2h}p_x^{02}$

Homework Solution 49.1.1

★★★★☆ Difficulty

(a) To find ${}_tp_x^{00}$, ${}_tp_x^{01}$, and ${}_tp_x^{02}$, we need to solve 3 differential equations with the help of the boundary conditions:

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})$$

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12})$$

$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{02}$$

$$\text{boundary conditions:} \quad {}_tp_x^{00} = 1, \quad {}_tp_x^{01} = {}_tp_x^{02} = 0$$

To understand the difficulty of solving these equations, notice that in the first equation, the term ${}_tp_x^{00}$ appears on both sides. In addition, the righthand side has a term ${}_tp_x^{01}$.

Like most other differential equations, these equation generally don't have exact solutions except under some simplifying assumptions. However, we can use numerical methods to approximate solutions to differential equations. There are many methods to approximate solutions to a differential equation. One of the oldest and easiest but probably the least efficient method was devised by Euler and is called the Euler method.

(b) This is the essence of the Euler method. Suppose we need to find the value of an unknown function $f(x)$ at $x = b$. We know the function's initial value $f(a)$. We also know the slope of $f(x)$ at any point. Then we can divide $[a, b]$ into n subintervals each of length $h = (b - a)/n$ and successively use the tangent line approximation to find $f(b)$.

$$f(a + h) \approx f(a) + hf'(a)$$

$$f(a + 2h) \approx f(a + h) + hf'(a + h)$$

$$\ldots$$

$$f(b) \approx f(b - h) + hf'(b - h)$$

The above method uses the forward recursion. The forward recursion works when you know the initial condition such as ${}_0p_x^{00} = 1$. However, in some cases we know only the ending condition. For example, for any insurance contract, at contract expiration, the policy value is always zero. From the ending zero policy value, we can use the Euler's method to find the beginning policy values. Here's the math for the backwards recursion.

$$f'(b) \approx \frac{f(b) - f(b - h)}{h}$$

$$f(b - h) \approx f(b) - f'(b)h$$

$$f(b-2h) \approx f(b-h) - f'(b-h)h$$

$$\cdots$$

(c)

$${}_hp_x^{00} \approx {}_0p_x^{00} + h\Big[{}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}(\mu_{x+t}^{01} + \mu_{x+t}^{02})\Big]_{t=0} = {}_0p_x^{00} + h\Big[{}_0p_x^{01}\mu_x^{10} - {}_0p_x^{00}(\mu_x^{01} + \mu_x^{02})\Big]$$

$${}_0p_x^{00} = 1, \quad {}_0p_x^{01} = 0$$

$$\mu_x^{01} + \mu_x^{02} = a_1 + b_1\exp(c_1x) + a_2 + b_2\exp(c_2x)$$

$$= 0.0004 + 3.4674 \times 10^{-6}e^{0.138155(60)} + 0.0005 + 7.5858 \times 10^{-5}e^{0.087498(60)} = 0.029158122$$

$$\Rightarrow {}_hp_x^{00} \approx 1 - \frac{1}{12}(\mu_x^{01} + \mu_x^{02}) = 1 - \frac{0.029158122}{12} = 0.997570$$

(d)

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12})$$

$$\Rightarrow {}_hp_x^{01} \approx {}_0p_x^{01} + h\Big[{}_0p_x^{00}\mu_x^{01} - {}_0p_x^{01}(\mu_x^{10} + \mu_x^{12})\Big] = h\mu_x^{01}$$

$$= \frac{0.0004 + 3.4674 \times 10^{-6}e^{0.138155(60)}}{12} = 0.001184$$

(e)

$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12}$$

$${}_hp_x^{02} \approx {}_0p_x^{02} + h\Big[{}_0p_x^{00}\mu_x^{02} + {}_0p_x^{01}\mu_x^{12}\Big] = h\mu_x^{02}$$

$$= \frac{0.014954241}{12} = 0.001246$$

Alternatively,

$${}_hp_x^{02} = 1 - ({}_hp_x^{00} + {}_hp_x^{01}) = 1 - (0.997570 + 0.001184) = 0.001246$$

(f)

$${}_{2h}p_x^{00} \approx {}_hp_x^{00} + h\Big[{}_hp_x^{01}\mu_{x+h}^{10} - {}_hp_x^{00}(\mu_{x+h}^{01} + \mu_{x+h}^{02})\Big]$$

$$= 0.997570 + \frac{0.001184 \times 0.001436372 - 0.997570(0.014363722 + 0.01506002)}{12} = 0.995124$$

(g)

$${}_{2h}p_x^{01} \approx {}_hp_x^{01} + h\Big[{}_hp_x^{00}\mu_{x+h}^{01} - {}_hp_x^{01}(\mu_{x+h}^{10} + \mu_{x+h}^{12})\Big]$$

$$= 0.001184 + \frac{0.997570 \times 0.014363722 - 0.001184(0.001436372 + 0.01506002)}{12} = 0.002376$$

(h)

$${}_{2h}p_x^{02} \approx {}_hp_x^{02} + h\Big[{}_hp_x^{00}\mu_{x+h}^{02} + {}_hp_x^{01}\mu_{x+h}^{12}\Big]$$

$$= 0.001246 + \frac{0.997570 \times 0.014363722 + 0.001184 \times 0.01506002}{12} = 0.002500$$

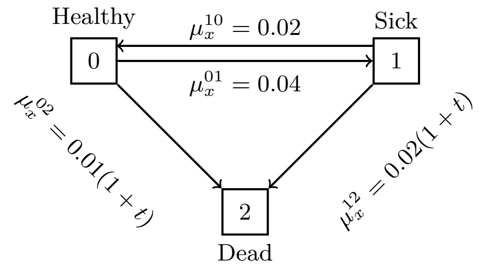
Alternatively,

$${}_{2h}p_x^{02} = 1 - ({}_{2h}p_x^{00} + {}_{2h}p_x^{01}) = 1 - (0.995124 + 0.002376) = 0.002500$$

By the way, the Euler method does not require you to divide the interval $[a, b]$ into subintervals of an equal length. However, equal length subintervals are often chosen for the ease of implementation.

Homework 49.1.2

You are given the following model:



Set $h = 0.1$. For $n = 1, 2, 3$, calculate the following probabilities:

(a) ${}_{nh}p_x^{00}$, ${}_{nh}p_x^{01}$, and ${}_{nh}p_x^{02}$

(b) ${}_{nh}p_x^{10}$, ${}_{nh}p_x^{11}$, and ${}_{nh}p_x^{12}$

(c) ${}_{nh}p_x^{20}$, ${}_{nh}p_x^{22}$, and ${}_{nh}p_x^{22}$

★★★★☆ Difficulty

t	μ_{x+t}^{01}	μ_{x+t}^{02}	μ_{x+t}^{10}	μ_{x+t}^{12}
0	0.04	0.010	0.02	0.020
h	0.04	0.011	0.02	0.022
$2h$	0.04	0.012	0.02	0.024

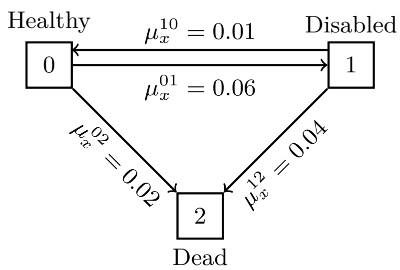
t	$\left(\frac{d}{dt}{}_tp_x^{00}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{01}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{02}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{10}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{11}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{12}\right)_t$
0	−0.05000000	0.04000000	0.01000000	0.02000000	−0.04000000	0.02000000
h	−0.05066500	0.03963200	0.01103300	0.01981800	−0.04175200	0.02193400
$2h$	−0.05131728	0.03924696	0.01207032	0.01962944	−0.04348102	0.02385158

t	${}_tp_x^{00}$	${}_tp_x^{01}$	${}_tp_x^{02}$	${}_tp_x^{10}$	${}_tp_x^{11}$	${}_tp_x^{02}$
0	1.0000000	0.0000000	0.0000000	0.0000000	1.0000000	0.0000000
h	0.9950000	0.0040000	0.0010000	0.0020000	0.9960000	0.0020000
$2h$	0.9899335	0.0079632	0.0021033	0.0039818	0.9918248	0.0041934
$3h$	0.9848018	0.0118879	0.0033103	0.0059447	0.9874767	0.0065786

$${}_tp_x^{20} = {}_tp_x^{21} = 0, \quad {}_tp_x^{22} = 1$$

Homework 49.1.3

The transition intensities are constants for all ages.



Set $h = \frac{1}{12}$. For $n = 1, 2, 3$, calculate the following probabilities:

(a) ${}_{nh}p_x^{00}$, ${}_{nh}p_x^{01}$, and ${}_{nh}p_x^{02}$

(b) ${}_{nh}p_x^{10}$, ${}_{nh}p_x^{11}$, and ${}_{nh}p_x^{12}$

★★★★☆ Difficulty

Homework Solution 49.1.2

μ_{x+t}^{01}	μ_{x+t}^{02}	μ_{x+t}^{10}	μ_{x+t}^{12}
0.06	0.02	0.01	0.04

t	$\left(\frac{d}{dt}{}_tp_x^{00}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{01}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{02}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{10}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{11}\right)_t$	$\left(\frac{d}{dt}{}_tp_x^{12}\right)_t$
0	−0.08000000	0.06000000	0.02000000	0.01000000	−0.05000000	0.04000000
h	−0.07941667	0.05935000	0.02006667	0.00989167	−0.04974167	0.03985000
$2h$	−0.07883776	0.05870563	0.02013214	0.00978427	−0.04948495	0.03970068

t	${}_tp_x^{00}$	${}_tp_x^{01}$	${}_tp_x^{02}$	${}_tp_x^{10}$	${}_tp_x^{11}$	${}_tp_x^{02}$
0	1.0000000	0.0000000	0.0000000	0.0000000	1.0000000	0.0000000
h	0.9933333	0.0050000	0.0016667	0.0008333	0.9958333	0.0033333
$2h$	0.9867153	0.0099458	0.0033389	0.0016576	0.9916882	0.0066542
$3h$	0.9801455	0.0148380	0.0050166	0.0024730	0.9875644	0.0099626

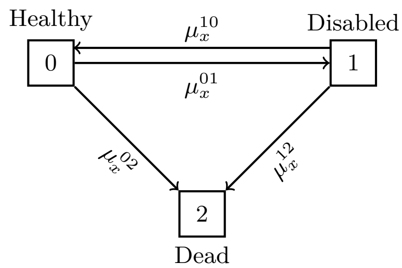
Chapter 50

Multiple state model: EPV of benefits, policy values, Thiele's differential equations

Example 50.0.1

An insurer issues a combined 5-year disability income and death benefit policy to a healthy life aged 65. You are given:

- $\mu_x^{01} = a_1 + b_1 \exp(c_1 x)$, $\mu_x^{10} = 0.1\mu_x^{01}$, $\mu_x^{02} = a_2 + b_2 \exp(c_2 x)$, $\mu_x^{12} = \mu_x^{02}$
- $a_1 = 0.0004$, $b_1 = 3 \times 10^{-6}$, $c_1 = 0.15$
- $a_2 = 0.0005$, $b_2 = 8 \times 10^{-5}$, $c_2 = 0.02$
- $\delta = 0.06$
- The premium is payable continuously at the rate of P per year while the insured is healthy.
- A benefit of \$50,000 per year is payable continuously while the insured is disabled.
- A death benefit of \$100,000 is payable immediately upon death.



(a) Write down the formula for the EPV of the premiums

(b) Write down the formula for the EPV of the disability benefit

(c) Write down the formula for the EPV of the death benefit

(d) Calculate P. Selective actuarial values:

- $\overline{a}^{00}_{65:\overline{5}|} = 3.684$
- $\int_0^5 e^{-\delta t} {}_tp_{65}^{00} \mu_{65+t}^{02} dt = 0.0579$
- $\int_0^5 e^{-\delta t} {}_tp_{65}^{01} \mu_{65+t}^{12} dt = 0.0102$
- $\overline{a}^{01}_{65:\overline{5}|} = 0.6008$

(e) Instead of paying \$50,000 while the insured disabled, the policy pays \$20,000 immediately upon disability. Write down the formula for the EPV of the disability benefit.

Solution 50.0.1

$$(a) \quad P\overline{a}^{00}_{65:\overline{5}|} = \int_0^5 {}_tp_{65}^{00} e^{-\delta t} dt$$

$$(b) \quad 50,000\overline{a}^{01}_{65:\overline{5}|} = 50,000 \int_0^5 {}_tp_{65}^{01} e^{-\delta t} dt$$

$$(c) \quad 100,000\overline{A}^{02}_{65:\overline{5}|} = 100,000 \int_0^5 e^{-\delta t} \left({}_tp_{65}^{00} \mu_{65+t}^{02} + {}_tp_{65}^{01} \mu_{65+t}^{12} \right) dt$$

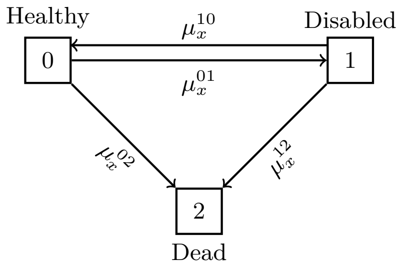
$$(d) \quad P = \frac{50,000\overline{a}^{01}_{65:\overline{5}|} + 100,000\overline{A}^{02}_{65:\overline{5}|}}{\overline{a}^{00}_{65:\overline{5}|}} = \frac{50,000 \times 0.6008 + 100,000(0.0579 + 0.0102)}{3.684} = 10,003$$

$$(e) \quad 20,000\overline{A}^{01}_{65:\overline{5}|} = 20,000 \int_0^5 e^{-\delta t} {}_tp_{65}^{00} \mu_{65+t}^{01} dt$$

Example 50.0.2

(Same model as that in the previous problem but the benefit structure is different) An insurer issues a combined 5-year disability and death benefit insurance policy to a healthy life aged 65. You are given:

- $\mu_x^{01} = a_1 + b_1 \exp(c_1 x)$, $\mu_x^{10} = 0.1\mu_x^{01}$, $\mu_x^{02} = a_2 + b_2 \exp(c_2 x)$, $\mu_x^{12} = \mu_x^{02}$
- $a_1 = 0.0004$, $b_1 = 3 \times 10^{-6}$, $c_1 = 0.15$
- $a_2 = 0.0005$, $b_2 = 8 \times 10^{-5}$, $c_2 = 0.02$
- $i = 0.06$
- A monthly premium P is payable in advance conditional on the life being healthy at the premium date.
- A benefit of \$50,000 per year is payable monthly in arrears while the life is disabled.
- A death benefit of \$100,000 is payable immediately upon death.



(a) Write down the formula for the EPV of the premiums.

(b) Of the two values $\ddot{a}^{(12)00}_{65:\overline{5}|}$ and $\overline{a}^{00}_{65:\overline{5}|}$, which one is bigger? Justify your answer.

(c) Write down the formula for the EPV of the disability benefit

(d) Write down the formula for the EPV of the death benefit.

(e) Write down the formula for P.

Solution 50.0.2

$$(a) \quad 12P\ddot{a}^{(12)00}_{65:\overline{5}|} = P\left(1 + {}_{1/12}p^{00}_{65}v^{1/12} + {}_{2/12}p^{00}_{65}v^{2/12} + {}_{3/12}p^{00}_{65}v^{3/12} + \ldots + {}_{5-1/12}p^{00}_{65}v^{5-1/12}\right)$$

$$(b) \quad \ddot{a}^{(12)00}_{65:\overline{5}|} > \overline{a}^{00}_{65:\overline{5}|}$$

More premium will be paid up front under $\ddot{a}^{(12)00}_{65:\overline{5}|}$, similar to the fact that $\ddot{a}^{00}_{65:\overline{5}|} > \overline{a}^{00}_{65:\overline{5}|}$.

$$(c) \quad 50{,}000a^{(12)01}_{65:\overline{5}|} = 50{,}000 \times \frac{1}{12}\left({}_{1/12}p^{01}_{65}v^{1/12} + {}_{2/12}p^{01}_{65}v^{2/12} + {}_{3/12}p^{01}_{65}v^{3/12} + \ldots + {}_{5}p^{01}_{65}v^{5}\right)$$

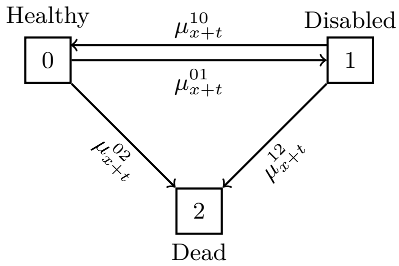
$$(d) \quad 100{,}000\overline{A}^{02}_{65:\overline{5}|} = 100{,}000\int_0^5 e^{-\delta t}\left({}_tp^{00}_{65}\mu^{02}_{65+t} + {}_tp^{01}_{65}\mu^{12}_{65+t}\right)dt$$

$$(e) \quad P = \frac{50{,}000a^{(12)01}_{65:\overline{5}|} + 100{,}000\overline{A}^{02}_{65:\overline{5}|}}{12\ddot{a}^{(12)00}_{65:\overline{5}|}}$$

50.1 Check your knowledge

Homework 50.1.1

An insurer issues a combined 10-year disability and death benefit policy to a healthy life aged 50. You are given:



(a) The product pays a continuous disability benefit at the rate of 50,000 per year while the insured is disabled, and pays a death benefit of 100,000 at the moment of death.

(b) Gross premium is payable continuously at the rate of P per year while the insured is healthy

(c) Premium expense is 2% of the gross premium. There are no other expenses.

(d) $\delta = 0.05$

(e) $\mu^{01}_{x+t} = 0.02$, $\mu^{10}_{x+t} = 0.01$, $\mu^{02}_{x+t} = 0.01 + 0.005t$, $\mu^{12}_{x+t} = 0.02 + 0.01t$

Selective actuarial values where $x = 50$ and $n = 10$:

k	$\overline{A}^{01}_{x+k:\overline{n-k\rvert}}$	$\overline{A}^{02}_{x+k:\overline{n-k\rvert}}$	$\overline{A}^{12}_{x+k:\overline{n-k\rvert}}$	$\overline{a}^{00}_{x+k:\overline{n-k\rvert}}$	$\overline{a}^{01}_{x+k:\overline{n-k\rvert}}$	$\overline{a}^{10}_{x+k:\overline{n-k\rvert}}$	$\overline{a}^{11}_{x+k:\overline{n-k\rvert}}$
0	0.12856	0.23701	0.37223	6.49636	0.50610	0.26205	6.14685
5	0.07652	0.19262	0.32982	3.84198	0.16607	0.08384	3.61567

(a) Calculate P.

(b) Actuary Mark decided to invent a new symbol $\overline{A}^{i\to j\to r}_{x:\overline{n\rvert}}$ to represent the APV of the benefit 1 payable immediately up on (x) moving from the state j to the state r during the first n years given that the insured is in the state i at time zero. Calculate the sum of $\overline{A}^{0\to 0\to 2}_{55:\overline{5\rvert}}$ and $\overline{A}^{0\to 1\to 2}_{55:\overline{5\rvert}}$ for this policy.

(c) Actuary Katie is a computation wizard and can calculate anything including $\overline{a}^{02}_{50+k:\overline{10-k\rvert}}$ and $\overline{a}^{12}_{50+k:\overline{10-k\rvert}}$ where $0 \le k \le 10$. Define these two symbols and explain whether it ever makes sense for her to calculate these two values for this policy.

(d) This is how Actuary Jeff explains the difference between $\overline{a}^{02}_{50:\overline{10\rvert}}$ and $\overline{a}^{12}_{50:\overline{10\rvert}}$: Both symbols represent the APV of the benefit 1 payable continuously while the insured is dead, with payments ceasing after first 10 years. However, $\overline{a}^{02}_{50:\overline{10\rvert}}$ is the APV of such benefits where the insured is healthy at death. In contrast, $\overline{a}^{12}_{50:\overline{10\rvert}}$ is the APV of such benefits where the insured is disabled at death. Is Jeff's explanation correct?

(e) Define the symbol $\overline{A}^{02}_{50+k:\overline{10-k\rvert}}$.

(f) Calculate the gross premium policy value ${}_5V^{(0)}$.

(g) Calculate the gross premium policy value ${}_5V^{(1)}$.

(h) Calculate the gross premium policy value ${}_5V^{(2)}$.

(i) Use the Euler's method and calculate the gross premium policy value two months before policy expiration given that the insured is healthy at that time.

(j) Use the Euler's method and calculate the gross premium policy value two months before policy expiration given that the insured is disabled at that time.

Homework Solution 50.1.1

★★★★☆ Difficulty

$$(a)\quad P = \frac{50,000\overline{a}^{01}_{50:\overline{10\rvert}} + 100,000\overline{A}^{02}_{50:\overline{10\rvert}}}{0.98\overline{a}^{00}_{50:\overline{10\rvert}}} = \frac{50,000 \times 0.50610 + 100,000 \times 0.23701}{0.98 \times 6.49636} = 7,697.62$$

$$(b)\quad \overline{A}^{0\to 0\to 2}_{55:\overline{5\rvert}} + \overline{A}^{0\to 1\to 2}_{55:\overline{5\rvert}} = \overline{A}^{02}_{55:\overline{5\rvert}} = 0.19262$$

$$(c)\quad \overline{a}^{02}_{50+k:\overline{10-k\rvert}} = \int_0^{10-k} e^{-\delta t}\,{}_tp^{02}_{50+k}dt, \quad \overline{a}^{12}_{50+k:\overline{10-k\rvert}} = \int_0^{10-k} e^{-\delta t}\,{}_tp^{12}_{50+k}dt$$

These two values will be useful if a benefit is paid continuously while the insured is dead given that the insured is currently healthy or disabled. Since no insurer will design such a policy, these two values will never be used in any useful actuarial calculations.

(d) Jeff's explanation is wrong. $\overline{a}^{02}_{50:\overline{10}|}$ includes the path $0 \to 2$ and the path $0 \to 1 \to 2$. For a payment to be made, the insured needs to be healthy at time zero and be dead at any time during the first ten years. The insured can be healthy or disabled at death.

Similarly, $\overline{a}^{12}_{50:\overline{10}|}$ includes the path $1 \to 2$ and the path $1 \to 0 \to 2$.

$$(e) \quad \overline{A}^{02}_{50+k:\overline{10-k}|} = \int_0^{10-k} \sum_{j\neq 2} e^{-\delta t}\, {}_tp^{0j}_{50+t}\, \mu^{j2}_{50+k} dt = \int_0^{10-k} e^{-\delta t}\, {}_tp^{00}_{50+t}\, \mu^{02}_{50+k} dt + \int_0^{10-k} e^{-\delta t}\, {}_tp^{01}_{50+t}\, \mu^{12}_{50+k} dt$$

$$(f) \quad {}_5V^{(0)} = 50,000\overline{a}^{01}_{55:\overline{5}|} + 100,000\overline{A}^{02}_{55:\overline{5}|} - 0.98P\overline{a}^{00}_{55:\overline{5}|}$$
$$= 50,000 \times 0.16607 + 100,000 \times 0.192617 - 0.98 \times 7,697.56 \times 3.84198 = 1,417$$

$$(g) \quad {}_5V^{(1)} = 50,000\overline{a}^{11}_{55:\overline{5}|} + 100,000\overline{A}^{12}_{55:\overline{5}|} - 0.98P\overline{a}^{10}_{55:\overline{5}|}$$
$$= 50,000 \times 3.61567 + 100,000 \times 0.32982 - 0.98 \times 7,697.56 \times 0.083841 = 213,133$$

$$(h) \quad {}_5V^{(2)} = 0 \quad \text{because the future payment is zero given that the insured is in the state 2}$$

$(i), (j)$

$$\text{for } 0 \le t \le 10$$
$$\frac{d}{dt}{}_tV^{(0)} = {}_tV^{(0)}\delta + 0.98P - \mu^{01}_{50+t}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{50+t}\left(100,000 - {}_tV^{(0)}\right)$$
$$\frac{d}{dt}{}_tV^{(1)} = {}_tV^{(1)}\delta - 50,000 - \mu^{10}_{50+t}\left({}_tV^{(0)} - {}_tV^{(1)}\right) - \mu^{12}_{50+t}\left(100,000 - {}_tV^{(1)}\right)$$
$${}_{10}V^{(0)} = {}_{10}V^{(1)} = 0$$

$$\left(\frac{d}{dt}{}_tV^{(0)}\right)_{t=10} = 0\delta + 0.98P - \mu^{01}_{50+10}(0-0) - \mu^{02}_{50+10}(100,000-0)$$
$$= 0.98P - \mu^{02}_{50+10}100,000 = 0.98 \times 7,697.56 - (0.01 + 0.005 \times 10)100,000 = 1543.6088$$

$$\left(\frac{d}{dt}{}_tV^{(0)}\right)_{t=10} \approx \frac{{}_{10}V^{(0)} - {}_{10-1/12}V^{(0)}}{1/12} = \frac{0 - {}_{10-1/12}V^{(0)}}{1/12} = 1543.6088$$
$$\Rightarrow \quad {}_{10-1/12}V^{(0)} \approx -128.64$$

$$\left(\frac{d}{dt}{}_tV^{(1)}\right)_{t=10} = 0\delta - 50,000 - \mu^{10}_{50+10}(0-0) - \mu^{12}_{50+10}(100,000-0)$$
$$= -50,000 - (0.02 + 0.01 \times 10)100,000 = -62,000.00$$

$$\left(\frac{d}{dt}{}_tV^{(1)}\right)_{t=10} \approx \frac{{}_{10}V^{(1)} - {}_{10-1/12}V^{(1)}}{1/12} = \frac{0 - {}_{10-1/12}V^{(1)}}{1/12} = -62,000.00$$
$$\Rightarrow \quad {}_{10-1/12}V^{(1)} \approx 5,166.67$$

$$\left(\frac{d}{dt}{}_tV^{(0)}\right)_{t=10-1/12}$$
$$= {}_{10-1/12}V^{(0)}\delta + 0.98P - \mu^{01}_{50+10-1/12}\left({}_{10-1/12}V^{(1)} - {}_{10-1/12}V^{(0)}\right) - \mu^{02}_{50+10-1/12}\left(100,000 - {}_{10-1/12}V^{(0)}\right)$$
$$= -128.64 \times 0.0 + 0.98 \times 7,697.56 - 0.02(5,166.67 - -128.64) - \left(0.01 + 0.005(10 - 1/12)\right)(100,000 - -128.64)$$
$$= 1,465.34$$

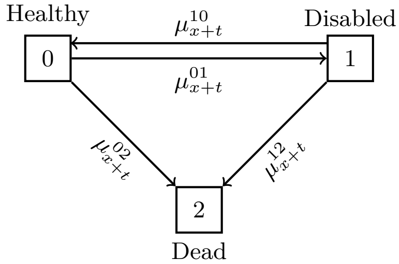
$$\left(\frac{d}{dt}{}_tV^{(0)}\right)_{t=10-1/12} \approx \frac{{}_{10-1/12}V^{(0)} - {}_{10-2/12}V^{(0)}}{1/12} = \frac{-128.64 - {}_{10-2/12}V^{(0)}}{1/12} = 1,465.34$$
$$\Rightarrow \quad {}_{10-2/12}V^{(0)} \approx -250.75$$

$$\left(\frac{d}{dt}{}_tV^{(1)}\right)_{t=10-1/12}$$
$$= {}_{10-1/12}V^{(1)}\delta - 50,000 - \mu^{10}_{50+10-1/12}\left({}_{10-1/12}V^{(0)} - {}_{10-1/12}V^{(1)}\right) - \mu^{12}_{50+10-1/12}\left(100,000 - {}_{10-1/12}V^{(1)}\right)$$
$$= 5,166.67 \times 0.05 - 50,000 - 0.01(-128.64 - 5,166.67) - \left(0.02 + 0.01(10 - 1/12)\right)(100,000 - 5,166.67)$$
$$= -60,989.69$$

$$\left(\frac{d}{dt}{}_tV^{(1)}\right)_{t=10-1/12} \approx \frac{{}_{10-1/12}V^{(1)} - {}_{10-2/12}V^{(1)}}{1/12} = \frac{5,166.67 - {}_{10-2/12}V^{(1)}}{1/12} = -60,989.69$$
$$\Rightarrow \quad {}_{10-2/12}V^{(1)} \approx 10,249.14$$

Homework 50.1.2

An insurer issues a combined 5-year disability and death benefit policy to a healthy life aged 60. You are given:



(a) Disability benefit: payable continuously at the rate of 75,000 per year while the insured is disabled

(b) Death benefit: 100,000 at the moment of death if the insured is healthy at death and 25,000 at the moment of death if the insured is disabled at death

(c) Gross premium: payable continuously at the rate of P per year while the insured is healthy

(d) Premium expenses: 5% of the premium incurred continuously.

(e) Death claim cost: 200 per claim.

(f) Maintenance expense while the insured is disabled: at the rate of 300 per year incurred continuously

(g) $\delta = 0.06$

(h) $\mu_{x+t}^{01} = 0.04$, $\mu_{x+t}^{02} = 0.01 + 0.02t$, $\mu_{x+t}^{10} = 0.02$, $\mu_{x+t}^{12} = 0.02 + 0.02t$

Selective actuarial values where $x = 60$ and $n = 5$:

k	$\overline{A}_{x+k:\overline{n-k}\rceil}^{01}$	$\overline{A}_{x+k:\overline{n-k}\rceil}^{02}$	$\overline{A}_{x+k:\overline{n-k}\rceil}^{12}$	$\overline{a}_{x+k:\overline{n-k}\rceil}^{00}$	$\overline{a}_{x+k:\overline{n-k}\rceil}^{01}$	$\overline{a}_{x+k:\overline{n-k}\rceil}^{10}$	$\overline{a}_{x+k:\overline{n-k}\rceil}^{11}$	$\int_0^{5-k} e^{-\delta t}{}_tp_{x+k}^{01}\mu_{x+k+t}^{12}dt$
0	0.14257	0.21562	0.24328	3.62381	0.32389	0.16194	3.74268	0.02564
3	0.06723	0.15422	0.16911	1.68923	0.06616	0.03308	1.76286	0.00644

You are also given: $\int_0^2 e^{-\delta t}{}_tp_{63}^{11}\mu_{63+t}^{12}dt = 0.16620$

(a) Show that $P = 12,800$ to the nearest of 50.

(b) Show that ${}_3V^{(0)} = -600$ to the nearest of 10.

(c) Calculate ${}_3V^{(1)}$.

(d) Calculate ${}_3V^{(2)}$.

(e) Write down the Thiele's differential equation for ${}_tV^{(0)}$ and for ${}_tV^{(1)}$ and the associated boundary conditions.

(f) Use the Euler's method and derive the backwards recursive formula for ${}_{t-h}V^{(0)}$ as a function of ${}_tV^{(0)}$; derive the backwards recursive formula for ${}_{t-h}V^{(1)}$ as a function of ${}_tV^{(1)}$.

(g) Use the backwards recursive formulas derived above and calculate ${}_{5-2/12}V^{(0)}$ and ${}_{5-2/12}V^{(1)}$ (e.g. the policy values two months before policy expiration).

Homework Solution 50.1.2

★★★★★ Difficulty

$$(a) \quad P = \frac{\text{PVFB}}{0.95\overline{a}_{60:\overline{5}\rceil}^{00}}$$

$$\text{PVFB} = 75,300\overline{a}_{60:\overline{5}\rceil}^{01} + 100,200\int_0^5 e^{-\delta t}{}_tp_{60}^{00}\mu_{60+t}^{02}dt + 25,200\int_0^5 e^{-\delta t}{}_tp_{60}^{01}\mu_{60+t}^{12}dt$$

$$= 75,300\overline{a}_{60:\overline{5}\rceil}^{01} + 100,200\left(\int_0^5 e^{-\delta t}{}_tp_{60}^{00}\mu_{60+t}^{02}dt + \int_0^5 e^{-\delta t}{}_tp_{60}^{01}\mu_{60+t}^{12}dt\right) - 75,000\int_0^5 e^{-\delta t}{}_tp_{60}^{01}\mu_{60+t}^{12}dt$$

$$= 75,300\overline{a}_{60:\overline{5}\rceil}^{01} + 100,200\overline{A}_{60:\overline{5}\rceil}^{02} - 75,000\int_0^5 e^{-\delta t}{}_tp_{60}^{01}\mu_{60+t}^{12}dt$$

$$= 75,300 \times 0.32389 + 100,200 \times 0.21562 - 75,000 \times 0.02564 = 44,071$$

$$P = \frac{44,071}{0.95\overline{a}_{60:\overline{5}\rceil}^{00}} = \frac{44,071}{0.95 \times 3.62381} = 12,802$$

$$(b)\quad {}_3V^{(0)} = 75,300\overline{a}^{01}_{63:\overline{2}|} + 100,200\overline{A}^{02}_{63:\overline{2}|} - 75,000\int_0^5 e^{-\delta t}{}_tp^{01}_{63}\mu^{12}_{63+t}dt - 0.95P\overline{a}^{00}_{63:\overline{2}|}$$

$$= 75,300\times 0.06616 + 100,200\times 0.15422 - 75,000\times 0.00644 - 0.95\times 12,802\times 1.68923 = -593$$

$$(c)\quad {}_3V^{(1)} = 75,300\overline{a}^{11}_{63:\overline{2}|} + 100,200\int_0^2 e^{-\delta t}{}_tp^{10}_{63}\mu^{02}_{63+t}dt + 25,200\int_0^5 e^{-\delta t}{}_tp^{11}_{63}\mu^{12}_{63+t}dt - 0.95P\overline{a}^{10}_{63:\overline{2}|}$$

$$= 75,300\overline{a}^{11}_{63:\overline{2}|} + 100,200\int_0^2 e^{-\delta t}{}_tp^{10}_{63}\mu^{02}_{63+t}dt + (100,200-75,000)\int_0^5 e^{-\delta t}{}_tp^{11}_{63}\mu^{12}_{63+t}dt - 0.95P\overline{a}^{10}_{63:\overline{2}|}$$

$$= 75,300\overline{a}^{11}_{63:\overline{2}|} + 100,200\overline{A}^{12}_{63:\overline{2}|} - 75,000\int_0^5 e^{-\delta t}{}_tp^{11}_{63}\mu^{12}_{63+t}dt - 0.95P\overline{a}^{10}_{63:\overline{2}|}$$

$$= 75,300\times 1.76286 + 100,200\times 0.16911 - 75,000\times 0.16620 - 0.95\times 12,802\times 0.03308 = 136,821$$

$$(d)\quad {}_3V^{(2)} = 0 \quad \text{because the future payment is zero given that the insured is in the state 2}$$

(e)

$$\text{for } 0\le t\le 5$$

$$\frac{d}{dt}{}_tV^{(0)} = {}_tV^{(0)}\delta + 0.95P - \mu^{01}_{60+t}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu^{02}_{60+t}\left(100,200 - {}_tV^{(0)}\right)$$

$$\frac{d}{dt}{}_tV^{(1)} = {}_tV^{(1)}\delta - 75,300 - \mu^{10}_{60+t}\left({}_tV^{(0)} - {}_tV^{(1)}\right) - \mu^{12}_{60+t}\left(25,200 - {}_tV^{(1)}\right)$$

$${}_5V^{(0)} = {}_5V^{(1)} = 0$$

$$(f)\quad \frac{d}{dt}{}_tV^{(0)} \approx \frac{{}_tV^{(0)} - {}_{t-h}V^{(0)}}{h},\quad \frac{d}{dt}{}_tV^{(1)} \approx \frac{{}_tV^{(1)} - {}_{t-h}V^{(0)}}{h}$$

$$\Rightarrow\quad {}_{t-h}V^{(0)} \approx {}_tV^{(0)} - h\frac{d}{dt}{}_tV^{(0)},\quad {}_{t-h}V^{(1)} \approx {}_tV^{(1)} - h\frac{d}{dt}{}_tV^{(1)}$$

(g)

$$\left(\frac{d}{dt}{}_tV^{(0)}\right)_{t=5} = {}_5V^{(0)}\delta + 0.95P - \mu^{01}_{60+5}\left({}_5V^{(1)} - {}_5V^{(0)}\right) - \mu^{02}_{60+5}\left(100,200 - {}_5V^{(0)}\right)$$

$$= 0\delta + 0.95\times 12,802 - \mu^{01}_{60+5}(0-0) - (0.01+0.02\times 5)(100,200-0) = 1,140$$

$$\left(\frac{d}{dt}{}_tV^{(1)}\right)_{t=5} = {}_5V^{(1)}\delta - 75,300 - \mu^{10}_{60+5}\left({}_5V^{(0)} - {}_5V^{(1)}\right) - \mu^{12}_{60+5}\left(25,200 - {}_5V^{(1)}\right)$$

$$= 0\delta - 75,300 - \mu^{10}_{60+5}(0-0) - (0.02+0.02\times 5)(25,200-0) = -78,324$$

$$\Rightarrow\quad {}_{5-1/12}V^{(0)} \approx {}_5V^{(0)} - h\left(\frac{d}{dt}{}_tV^{(0)}\right)_{t=5} = 0 - \frac{1}{12}(1,140) = -95$$

$$\Rightarrow\quad {}_{5-1/12}V^{(1)} \approx {}_5V^{(1)} - h\left(\frac{d}{dt}{}_tV^{(1)}\right)_{t=5} = 0 - \frac{1}{12}(-78,324) = 6,527$$

$$\left(\frac{d}{dt}{}_tV^{(0)}\right)_{t=5-1/12} = {}_{5-1/12}V^{(0)}\delta + 0.95P - \mu^{01}_{60+5-1/12}\left({}_{5-1/12}V^{(1)} - {}_{5-1/12}V^{(0)}\right) - \mu^{02}_{60+5-1/12}\left(100,200 - {}_{5-1/12}V^{(0)}\right)$$

$$= -95\times 0.06 + 0.95\times 12,802 - 0.04(6,527 - -95) - \big(0.01 + 0.02(5-1/12)\big)(100,200 - -95) = 1,026$$

$$\left(\frac{d}{dt}{}_tV^{(1)}\right)_{t=5-1/12} = {}_{5-1/12}V^{(1)}\delta - 75,300 - \mu^{10}_{60+5-1/12}\left({}_{5-1/12}V^{(0)} - {}_{5-1/12}V^{(1)}\right) - \mu^{12}_{60+5-1/12}\left(25,200 - {}_{5-1/12}V^{(1)}\right)$$

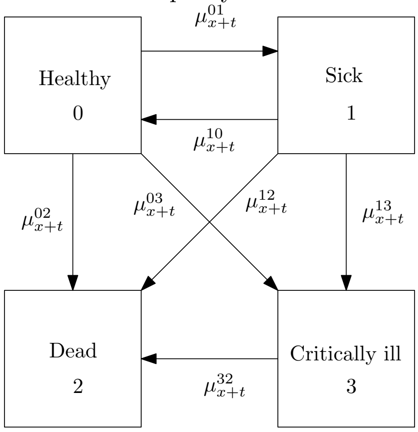
$$= 6,527\times 0.06 - 75,300 - 0.02(-95 - 6,527) - \big(0.02 + 0.02\times(5-1/12)\big)(25,200 - 6,527) = -76,986$$

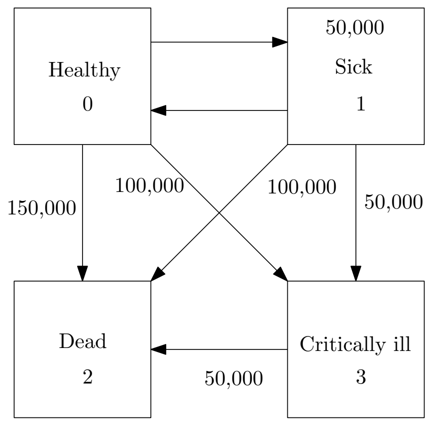
$$\Rightarrow\quad {}_{5-2/12}V^{(0)} \approx {}_{5-1/12}V^{(0)} - h\left(\frac{d}{dt}{}_tV^{(0)}\right)_{t=5-1/12} = -95 - \frac{1}{12}(1,026) = -181$$

$$\Rightarrow\quad {}_{5-2/12}V^{(1)} \approx {}_{5-1/12}V^{(1)} - h\left(\frac{d}{dt}{}_tV^{(1)}\right)_{t=5-1/12} = 6,527 - \frac{1}{12}(-76,986) = 12,943$$

Homework 50.1.3

(Model parameters are from the AMLCR Exercise 8.7) A combined 10-year term disability income, critical illness benefit, and death benefit policy is issued to a healthy life age 60.





You are given:

- $\mu^{01}_x = a_1 + b_1 \exp\{c_1 x\}$, $\mu^{02}_x = a_2 + b_2 \exp\{c_2 x\}$
- $\mu^{12}_x = \mu^{02}_x$, $\mu^{32}_x = 1.2\mu^{02}_x$
- $\mu^{10}_x = 0.1\mu^{01}_x$, $\mu^{03}_x = 0.05\mu^{01}_x$, $\mu^{13}_x = \mu^{03}_x$
- $a_1 = 4 \times 10^{-4}$, $b_1 = 3.5 \times 10^{-6}$, $c_1 = 0.14$
- $a_2 = 5 \times 10^{-4}$, $b_2 = 7.6 \times 10^{-5}$, $c_2 = 0.09$
- $\delta = 6\%$
- Net premium: payable continuously at the rate of P per year while the insured is healthy.

Benefits:

- Death benefit payable immediately upon death: 150,000 if the insured is healthy at death, 100,000 if the insured is disabled (e.g. sick) at death, and 50,000 if the insured is critically ill at death.
- Disability benefit: 50,000 per year payable continuously where the insured is disabled.
- Critical illness benefit payable immediately upon becoming critically ill: 100,000 if the insured is healthy before turning critically ill and 50,000 if the insured is sick before turning critically ill.

Selective actuarial values for $x = 60$ and $n = 10$:

k	$\overline{A}^{01}_{x+k:\overline{n-k}\rvert}$	$\overline{A}^{02}_{x+k:\overline{n-k}\rvert}$	$\overline{A}^{03}_{x+k:\overline{n-k}\rvert}$	$\overline{A}^{12}_{x+k:\overline{n-k}\rvert}$	$\overline{a}^{00}_{x+k:\overline{n-k}\rvert}$	$\overline{a}^{01}_{x+k:\overline{n-k}\rvert}$	$\overline{a}^{10}_{x+k:\overline{n-k}\rvert}$	$\overline{a}^{11}_{x+k:\overline{n-k}\rvert}$
0	0.19161	0.18605	0.01105	0.18457	6.39703	0.72502	0.07250	7.07677

k	$\int_0^{n-k} e^{-\delta t}\,{}_tp^{01}_x\,\mu^{12}_{x+t}dt$	$\int_0^{n-k} e^{-\delta t}\,{}_tp^{01}_x\,\mu^{13}_{x+t}dt$	$\int_0^{n-k} e^{-\delta t}\,{}_tp^{03}_x\,\mu^{32}_{x+t}dt$
0	0.02235	0.00147	0.00148

(a) Show that $P = 10,000$ to the nearest of 10.

(b) Derive the formula for ${}_tV^{(0)}$ where $0 < t < 10$.

(c) Derive the formula for ${}_tV^{(1)}$ where $0 < t < 10$.

(d) Derive the formula for ${}_tV^{(3)}$ where $0 < t < 10$.

(e) Use general reasoning to derive the formula for $\dfrac{d}{dt}{}_tV^{(0)}$ and $\dfrac{d}{dt}{}_tV^{(1)}$ where $0 < t < 10$. Rigorous proof is not expected.

(f) Estimate the net premium P. You are given:

- ${}_0V^{(0)} = 1,585.94$ if $P = 9,800$
- ${}_0V^{(0)} = -327.15$ if $P = 10,100$

(g) Re-do Part (e) assuming that the insurer charges a single net premium at issue.

Homework Solution 50.1.3

★★★★★ Difficulty

(a)

$$\text{PVFB death} = 150,000 \int_0^{10} e^{-\delta t}\,{}_tp^{00}_{60}\mu^{02}_{60+t}dt + 100,000 \int_0^{10} e^{-\delta t}\,{}_tp^{01}_{60}\mu^{12}_{60+t}dt + 50,000 \int_0^{10} e^{-\delta t}\,{}_tp^{03}_{60}\mu^{32}_{60+t}dt$$

$$\overline{A}^{02}_{60:\overline{5}\rvert} = \int_0^{10} e^{-\delta t}\,{}_tp^{00}_{60}\mu^{02}_{60+t}dt + \int_0^{10} e^{-\delta t}\,{}_tp^{01}_{60}\mu^{12}_{60+t}dt + \int_0^{10} e^{-\delta t}\,{}_tp^{03}_{60}\mu^{32}_{60+t}dt = 0.18605$$

$$\int_0^{10} e^{-\delta t}\,{}_tp^{01}_{60}\mu^{12}_{60+t}dt = 0.02235, \qquad \int_0^{10} e^{-\delta t}\,{}_tp^{03}_{60}\mu^{32}_{60+t}dt = 0.00148$$

$$\text{PVFB death} = 150,000 \times 0.18605 - 50,000 \times 0.02235 - 100,000 \times 0.00148 = 26,642$$

$$\text{PVFB disability} = 50,000\overline{a}^{01}_{60:\overline{10|}} = 50,000 \times 0.72502 = 36,251$$

$$\text{PVFB CI} = 100,000\int_0^{10} e^{-\delta t}\,{}_tp^{00}_{60}\mu^{03}_{60+t}dt + 50,000\int_0^{10} e^{-\delta t}\,{}_tp^{01}_{60}\mu^{13}_{60+t}dt$$

$$= 100,000\overline{A}^{03}_{60:\overline{10|}} - 50,000\int_0^{10} e^{-\delta t}\,{}_tp^{01}_{60}\mu^{13}_{60+t}dt = 100,000 \times 0.01105 - 50,000 \times 0.00147 = 1,032$$

$$\text{PVFB} = 26,642 + 36,251 + 1,032 = 63,925$$

$$P = \frac{\text{PVFB}}{\overline{a}^{00}_{60:\overline{10|}}} = \frac{63,925}{6.39703} = 9,993$$

(*b*)

$${}_kV^{(0)} = \text{PVFB death} + \text{PVFB disability} + \text{PVFB CI} - \text{PVFPrem}$$

conditional on the insured being in the state 0 at time k

$${}_kV^{(0)} = 150,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{00}_{60+k}\mu^{02}_{60+k+t}dt + 100,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{01}_{60+k}\mu^{12}_{60+k+t}dt + 50,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{03}_{60+k}\mu^{32}_{60+k+t}dt$$

$$+50,000\overline{a}^{01}_{60+k:\overline{10-k|}}$$

$$+100,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{00}_{60+k}\mu^{03}_{60+k+t}dt + 50,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{01}_{60+k}\mu^{13}_{60+k+t}dt$$

$$-P\overline{a}^{00}_{60+k:\overline{10-k|}}$$

(*c*)

$${}_kV^{(1)} = \text{PVFB death} + \text{PVFB disability} + \text{PVFB CI} - \text{PVFPrem}$$

conditional on the insured being in the state 1 at time k

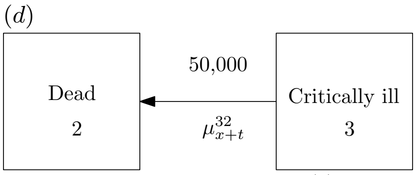
$${}_kV^{(1)} = 150,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{10}_{60+k}\mu^{02}_{60+k+t}dt + 100,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{11}_{60+k}\mu^{12}_{60+k+t}dt + 50,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{13}_{60+k}\mu^{32}_{60+k+t}dt$$

$$+50,000\overline{a}^{11}_{60+k:\overline{10-k|}}$$

$$+100,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{10}_{60+k}\mu^{03}_{60+k+t}dt + 50,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{11}_{60+k}\mu^{13}_{60+k+t}dt$$

$$-P\overline{a}^{10}_{60+k:\overline{10-k|}}$$

(*d*)



$${}_kV^{(3)} = 50,000\overline{A}^{32}_{60+k:\overline{10-k|}} = 50,000\int_0^{10-k} e^{-\delta t}\,{}_tp^{33}_{60+k}\mu^{32}_{60+k+t}dt$$

(*f*) It's useful to think of ${}_tV^{(0)}$ as the savings account value at t per insured who is in the state 0. Future premiums from the insureds, given that these insureds are in the state 0 at t, are deposited into this account; future claims from the insureds, given that these insureds are in the state 0 at t, are paid out from this account. At any time before contract expiration, the expected account value should exactly fund the expected future claims.

Consider what happens during a tiny interval $[h, t+h]$ where $t < t+h < 10$. Premiums received and the interest earned will increase the savings account value to:

$${}_tV^{(0)}e^{\delta h} + P\overline{s}_h = {}_tV^{(0)}(1+\delta h) + Ph + o(h)$$

The liabilities at $t+h$ consists of the policy value ${}_{t+h}V^{(0)}$ and possible extra amounts of

- ${}_{t+h}V^{(1)} - {}_{t+h}V^{(0)}$ if the insured travels along the path $0 \to 1$, the probability of which is $h\mu^{01}_{60+t} + o(h)$
- $150,000 - {}_{t+h}V^{(0)}$ if the insured travels along the path $0 \to 2$, the probability of which is $h\mu^{02}_{60+t} + o(h)$
- $100,000 - {}_{t+h}V^{(0)}$ if the insured travels along the path $0 \to 3$, the probability of which is $h\mu^{03}_{60+t} + o(h)$

$${}_tV^{(0)}(1+\delta h)+Ph = {}_{t+h}V^{(0)}+h\mu^{01}_{60+t}\left({}_{t+h}V^{(1)}-{}_{t+h}V^{(0)}\right)+h\mu^{02}_{60+t}\left(150,000-{}_{t+h}V^{(0)}\right)+h\mu^{03}_{60+t}\left(100,000-{}_{t+h}V^{(0)}\right)+o(h)$$

Rearrange the above formula, divide by h, and let $h \to 0$:

$$\frac{d}{dt}{}_tV^{(0)} = {}_tV^{(0)}\delta + P - \mu^{01}_{60+t}\left({}_tV^{(1)}-{}_tV^{(0)}\right)-\mu^{02}_{60+t}\left(150,000-{}_tV^{(0)}\right)-\mu^{03}_{60+t}\left(100,000-{}_tV^{(0)}\right)$$

Follow the same reasoning:

$$\frac{d}{dt}{}_tV^{(1)} = {}_tV^{(1)}\delta - 50,000 - \mu^{10}_{60+t}\left({}_tV^{(0)}-{}_tV^{(1)}\right)-\mu^{12}_{60+t}\left(100,000-{}_tV^{(1)}\right)-\mu^{13}_{60+t}\left(50,000-{}_tV^{(1)}\right)$$

(e) We'll use linear interpolation to find P such that ${}_0V^{(0)}=0$.

$$P \approx 9,800+\frac{0-1,585.94}{-327.15-1,585.94}\times(10,100-9,800)=10,049$$

This is different from the premium in Part (a) because the two methods are different.

(f)

$$\frac{d}{dt}{}_tV^{(0)} = {}_tV^{(0)}\delta - \mu^{01}_{60+t}\left({}_tV^{(1)}-{}_tV^{(0)}\right)-\mu^{02}_{60+t}\left(150,000-{}_tV^{(0)}\right)-\mu^{03}_{60+t}\left(100,000-{}_tV^{(0)}\right)$$

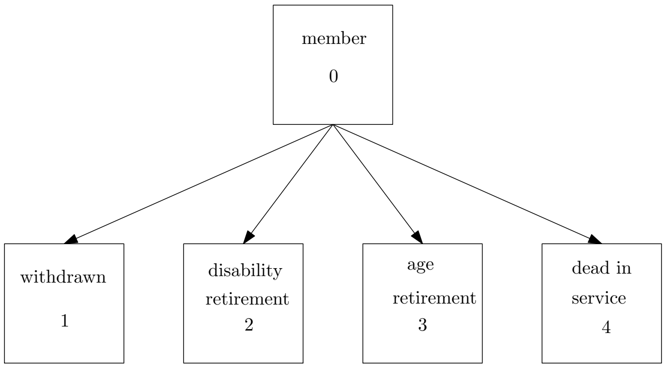
The formula for $\frac{d}{dt}{}_tV^{(1)}$ remains the same.

Chapter 51

Multiple state: age retirement, disability, withdrawal

Example 51.0.1

You are given the following multiple state model:



(i)
$$\mu_x^{01} = \begin{cases} 0.08 & \text{for } 40 \leq x < 50 \\ 0.04 & \text{for } 50 \leq x < 60 \\ 0.00 & \text{for } x \geq 60 \end{cases}$$

(ii) $\mu_x^{02} = 0.002$

(iii)
$$\mu_x^{03} = \begin{cases} 0.0 & \text{for } x < 60 \\ 0.1 & \text{for } 60 \leq x < 65 \end{cases}$$

(iv) 20% of the members surviving in employment to age 60 retire at that time.

(v) 100% of the members surviving in employment to age 65 retire at that time.

(vi) $\mu_x^{04} = 0.005$

(a) For each mode of exit, calculate the probability that a member aged 40 exits employment by that mode.

(b) Calculate the probability that a member aged 40 will retire at age 65.

Solution 51.0.1

	μ^{01}_{40+t}	μ^{02}_{40+t}	μ^{03}_{40+t}	μ^{04}_{40+t}	Total	${}_tp^{00}_{40}$
$0 \leq t < 10$	0.08	0.002	0.0	0.005	0.087	$e^{-0.087t}$
$10 \leq t < 20$	0.04	0.002	0.0	0.005	0.047	$e^{-0.087\times 10}e^{-0.047(t-10)}$
$t = 20^+$						$e^{-0.087\times 10}e^{-0.047\times 10}0.8$
$20^+ < t < 25$	0.00	0.002	0.1	0.005	0.107	$e^{-0.087\times 10}e^{-0.047\times 10}0.8e^{-0.107(t-20)}$

(a)

$$P(\text{exit by force } j \text{ by age 50}) = {}_{10}p^{0j}_{40} = \int_0^{10} {}_tp^{00}_{40}\mu^{0j}_{40+t}dt = \int_0^{10} e^{-0.087t}\mu^{0j}_{40+t}dt = \frac{\mu^{0j}_{40+t}}{0.087}\left(1 - e^{-0.087\times 10}\right)$$

$$P(\text{exit by force } j \text{ after age 50 and before age 60}) = \int_{10}^{20} {}_tp^{00}_{40}\mu^{0j}_{40+t}dt = \int_{10}^{20} e^{-0.087\times 10}e^{-0.047(t-10)}\mu^{0j}_{40+t}dt$$

$$= e^{-0.087\times 10}\int_0^{10} e^{-0.047t}\mu^{0j}_{40+10+t}dt = e^{-0.087\times 10}\frac{\mu^{0j}_{40+10+t}}{0.047}\left(1 - e^{-0.047\times 10}\right)$$

$$P(\text{exit by force } j \text{ after age 60 and before age 65}) = \int_{20^+}^{25} {}_tp^{00}_{40}\mu^{02}_{40+t}dt$$

$$= \int_{20^+}^{25} e^{-0.087\times 10}e^{-0.047\times 10}0.8e^{-0.107(t-20)}\mu^{0j}_{40+t}dt = \int_0^{5} e^{-0.087\times 10}e^{-0.047\times 10}0.8e^{-0.107t}\mu^{0j}_{40+20+t}dt$$

$$= e^{-0.087\times 10}e^{-0.047\times 10} \times 0.8 \times \frac{\mu^{0j}_{40+20+t}}{0.107}\left(1 - e^{-0.107\times 5}\right)$$

$$P(\text{exit by force 1}) = \frac{0.08}{0.087}\left(1 - e^{-0.087\times 10}\right) + e^{-0.087\times 10}\frac{0.04}{0.047}\left(1 - e^{-0.047\times 10}\right) + e^{-0.087\times 10}e^{-0.047\times 10}\times 0.8\times\frac{0}{0.107}\left(1 - e^{-0.107\times 5}\right)$$

$$= \boxed{0.66800456}$$

$$P(\text{exit by force 2}) = \frac{0.002}{0.087}\left(1-e^{-0.087\times10}\right)+e^{-0.087\times10}\frac{0.002}{0.047}\left(1-e^{-0.047\times10}\right)+e^{-0.087\times10}e^{-0.047\times10}\times0.8\times\frac{0.002}{0.107}\left(1-e^{-0.107\times5}\right)$$
$$= \boxed{0.02166508}$$

$$P(\text{exit by force 4}) = \frac{0.005}{0.087}\left(1-e^{-0.087\times10}\right)+e^{-0.087\times10}\frac{0.005}{0.047}\left(1-e^{-0.047\times10}\right)+e^{-0.087\times10}e^{-0.047\times10}\times0.8\times\frac{0.005}{0.107}\left(1-e^{-0.107\times5}\right)$$
$$= \frac{5}{2}\times P(\text{exit by force 4}) = \boxed{0.05416271}$$

$$P(\text{exit by force 3 excluding point decrement at age 60 and age 65})$$
$$= \frac{0}{0.087}\left(1-e^{-0.087\times10}\right)+e^{-0.087\times10}\frac{0}{0.047}\left(1-e^{-0.047\times10}\right)+e^{-0.087\times10}e^{-0.047\times10}\times0.8\times\frac{0.1}{0.107}\left(1-e^{-0.107\times5}\right)$$
$$= 0.08111454$$

$$P(\text{retire at age 60}) = 0.2_{20-}p_{40}^{00} = e^{-0.087\times10}e^{-0.047\times10} = 0.05236913$$
$$P(\text{retire at age 65}) = {}_{25+}p_{40}^{00} = e^{-0.087\times10}e^{-0.047\times10}0.8e^{-0.107\times5} = 0.12268397$$

$$\Rightarrow \quad P(\text{exit by age retirement}) = 0.08111454+0.05236913+0.12268397 = \boxed{0.25616764}$$

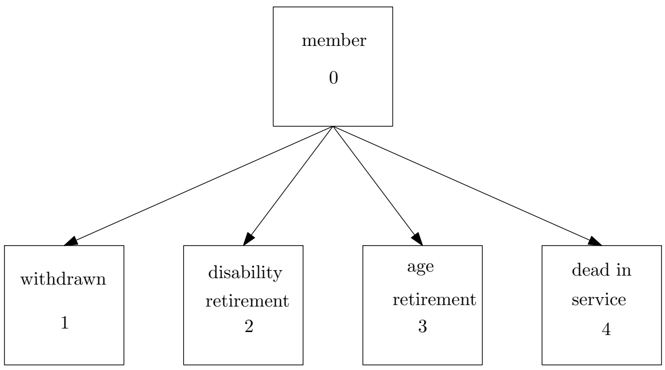
check total:
$$0.66800456+0.02166508+0.05416271+0.25616764 = 0.99999999 \approx 1 \text{ OK}$$

$$(b) \quad {}_{25-}p_{40}^{00} = e^{-0.087\times10}e^{-0.047\times10}0.8e^{-0.107(25-20)} = 0.12268397$$

51.1 Check your knowledge

Homework 51.1.1

You are given the following multiple state model:



(i)
$$\mu_x^{01} = \begin{cases} 0.06 & \text{for } 30 \le x < 45 \\ 0.02 & \text{for } 45 \le x < 60 \\ 0.00 & \text{for } x \ge 60 \end{cases}$$
$$\mu_x^{02} = \begin{cases} 0.002 & \text{for } 30 \le x < 45 \\ 0.004 & \text{for } 45 \le x < 60 \\ 0.006 & \text{for } x \ge 60 \end{cases}$$

(ii)
$$\mu_x^{03} = \begin{cases} 0.0 & \text{for } x < 60 \\ 0.2 & \text{for } 60 \le x < 65 \end{cases}$$

(iii) 40% of the members surviving in employment to age 60 retire at that time.

(iv) 100% of the members surviving in employment to age 65 retire at that time.

(v)
$$\mu_x^{04} = \begin{cases} 0.005 & \text{for } x < 60 \\ 0.010 & \text{for } 60 \le x < 65 \end{cases}$$

(a) For each mode of exit, calculate the probability that a member aged 30 exits employment by that mode.

(b) Calculate the probability that a member aged 30 will retire at age 65.

Homework Solution 51.1.1

★★★★☆ Difficulty

	μ_{30+t}^{01}	μ_{30+t}^{02}	μ_{30+t}^{03}	μ_{30+t}^{04}	Total	${}_tp_{30}^{00}$
$0 \le t < 15$	0.06	0.002	0.0	0.005	0.067	$e^{-0.067t}$
$15 \le t < 30$	0.02	0.004	0.0	0.005	0.029	$e^{-0.067\times15}e^{-0.029(t-15)}$
$t = 30^+$						$e^{-0.067\times15}e^{-0.029\times15}0.6$
$30^+ < t < 35$	0.00	0.006	0.2	0.010	0.216	$e^{-0.067\times15}e^{-0.029\times15}0.6e^{-0.216(t-30)}$

Homework Solution 51.1.2

★★★★☆ Difficulty

(a)

$$P(\text{exit by force } j \text{ by age } 45) = {}_{15}p_{30}^{0j} = \int_0^{15} {}_tp_{30}^{00}\mu_{30+t}^{0j}dt = \int_0^{15} e^{-0.067t}\mu_{30+t}^{0j}dt = \frac{\mu_{30+t}^{0j}}{0.067}\left(1-e^{-0.067\times 15}\right)$$

$$P(\text{exit by force } j \text{ after age 45 and before age 60}) = \int_{15}^{30} {}_tp_{30}^{00}\mu_{30+t}^{0j}dt = \int_{15}^{30} e^{-0.067\times 15}e^{-0.029(t-15)}\mu_{30+t}^{0j}dt$$

$$= e^{-0.067\times 15}\int_0^{15} e^{-0.029t}\mu_{30+15+t}^{0j}dt = e^{-0.067\times 15}\frac{\mu_{30+15+t}^{0j}}{0.029}\left(1-e^{-0.029\times 15}\right)$$

$$P(\text{exit by force } j \text{ after age 60 and before age 65}) = \int_{30^+}^{25} {}_tp_{30}^{00}\mu_{30+t}^{02}dt$$

$$= \int_{30^+}^{25} e^{-0.067\times 15}e^{-0.029\times 15}0.6e^{-0.216(t-30)}\mu_{30+t}^{0j}dt = \int_0^{5} e^{-0.067\times 15}e^{-0.029\times 15}0.6e^{-0.216t}\mu_{30+30+t}^{0j}dt$$

$$= e^{-0.067\times 15}e^{-0.029\times 15}\times 0.6\times\frac{\mu_{30+30+t}^{0j}}{0.216}\left(1-e^{-0.216\times 5}\right)$$

$$P(\text{exit by force 1}) = \frac{0.06}{0.067}\left(1-e^{-0.067\times 15}\right)+e^{-0.067\times 15}\frac{0.02}{0.029}\left(1-e^{-0.029\times 15}\right)+e^{-0.067\times 15}e^{-0.029\times 15}\times 0.6\times\frac{0}{0.216}\left(1-e^{-0.216\times 5}\right)$$

$$= \boxed{0.656767344}$$

$$P(\text{exit by force 2}) = \frac{0.002}{0.067}\left(1-e^{-0.067\times 15}\right)+e^{-0.067\times 15}\frac{0.004}{0.029}\left(1-e^{-0.029\times 15}\right)+e^{-0.067\times 15}e^{-0.029\times 15}\times 0.6\times\frac{0.006}{0.216}\left(1-e^{-0.216\times 5}\right)$$

$$= \boxed{0.039341068}$$

$$P(\text{exit by force 4}) = \frac{0.005}{0.067}\left(1-e^{-0.067\times 15}\right)+e^{-0.067\times 15}\frac{0.005}{0.029}\left(1-e^{-0.029\times 15}\right)+e^{-0.067\times 15}e^{-0.029\times 15}\times 0.6\times\frac{0.010}{0.216}\left(1-e^{-0.216\times 5}\right)$$

$$= \boxed{0.07391797}$$

$$P(\text{exit by force 3 excluding point decrement at age 60 and age 65})$$

$$= \frac{0}{0.067}\left(1-e^{-0.067\times 15}\right)+e^{-0.067\times 15}\frac{0}{0.029}\left(1-e^{-0.029\times 15}\right)+e^{-0.067\times 15}e^{-0.029\times 15}\times 0.6\times\frac{0.2}{0.216}\left(1-e^{-0.216\times 5}\right)$$
$$= 0.086926751$$

$$P(\text{retire at age 60}) = 0.4\,{}_{30^-}p_{30}^{00} = e^{-0.067\times 15}e^{-0.029\times 15} = 0.0947711$$
$$P(\text{retire at age 65}) = {}_{25^+}p_{30}^{00} = e^{-0.067\times 15}e^{-0.029\times 15}0.6e^{-0.216\times 5} = 0.04827576$$

$$\Rightarrow \quad P(\text{exit by age retirement}) = 0.086926751+0.0947711+0.04827576 = \boxed{0.22997361}$$

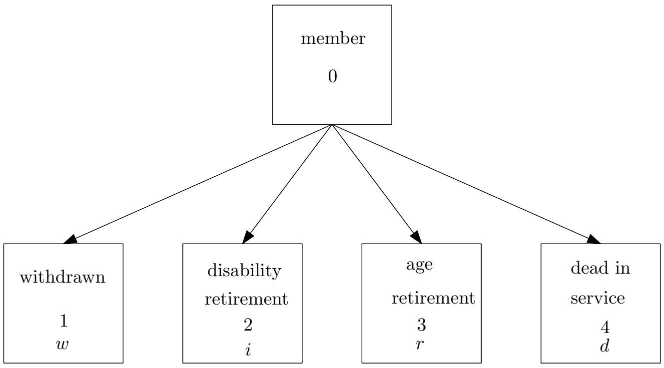
check total:

$$0.656767344+0.039341068+0.07391797+0.22997361 = 0.99999999\approx 1 \text{ OK}$$

$$(b)\quad {}_{35^-}p_{30}^{00} = e^{-0.067\times 15}e^{-0.029\times 15}0.6e^{-0.216(35-30)} = 0.04827576$$

Homework 51.1.2

You are given the following multiple state model:



t	${}_tp^{00}_{20+t}$	${}_tp^{0w}_{20+t}$	${}_tp^{0i}_{20+t}$	${}_tp^{0r}_{20+t}$	${}_tp^{0d}_{20+t}$
0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000
1	0.9037074	0.0951042	0.0009510	0.0000000	0.0002374
2	0.8166840	0.1810504	0.0018105	0.0000000	0.0004551
3	0.7380376	0.2587202	0.0025872	0.0000000	0.0006551
4	0.6669617	0.3289103	0.0032891	0.0000000	0.0008391
5	0.6027276	0.3923406	0.0039234	0.0000000	0.0010086

Use the radius $\ell_{20} = 1,000,000$ and construct the pension service table.

Homework Solution 51.1.3

★★★★☆ Difficulty

t	$\Delta_t p^{0w}_{20+t}$	$\Delta_t p^{0i}_{20+t}$	$\Delta_t p^{0r}_{20+t}$	$\Delta_t p^{0d}_{20+t}$
0				
1	0.0951042	0.0009510	0.0000000	0.0002374
2	0.0859462	0.0008595	0.0000000	0.0002177
3	0.0776698	0.0007767	0.0000000	0.0002000
4	0.0701901	0.0007019	0.0000000	0.0001840
5	0.0634303	0.0006343	0.0000000	0.0001695

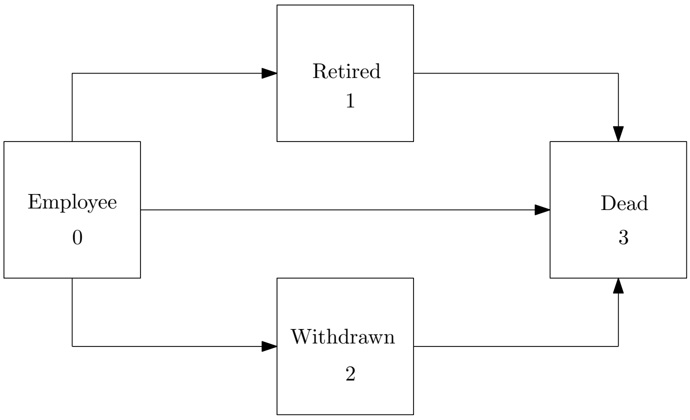
Example $0.0859462 = 0.1810504 - 0.0951042$

t	ℓ_x	w_x	i_x	r_x	d_x
20	1,000,000	95,104	951	0	237
21	903,707	85,946	860	0	218
22	816,684	77,670	777	0	200
23	738,038	70,190	702	0	184
24	666,962	63,430	634	0	170

Example $95,104 = 1,000,000 \times 0.0951042$, $\quad 951 = 1,000,000 \times 0.0009510$, $\quad 903,707 = 1,000,000 - (95,104 + 951 + 0 + 237)$

Homework 51.1.3

You are given the following withdrawal/retirement model:



You are also given:

(i) The force of mortality depends on the individual's age only and follows the Illustrative Life Table. That is, $\mu_x^{03} = \mu_x^{13} = \mu_x^{23} = \mu_x$, where μ_x is equal to the force of mortality in the Illustrative Life Table (ILT).

(ii)
$$\mu_x^{02} = \begin{cases} 0.002 & \text{if } x < 60 \\ 0.000 & \text{if } x \geq 60 \end{cases}$$

(iii) Employees retire exactly on age 60, 61, 62, 63, 64, or 65.

(iv) 20% of the employees reaching age 60, 61, 62, 63, and 64 retire at that age; 100% of the employees reaching age 65 retire at the age.

(v) Death benefit after retirement: 1 at the end of the year of death.

(vi) $i = 0.06$.

For an employee currently aged 50, derive the EPV of the post-retirement death benefit. Numerical calculations are not expected.

Homework Solution 51.1.4

★★★★★ Difficulty

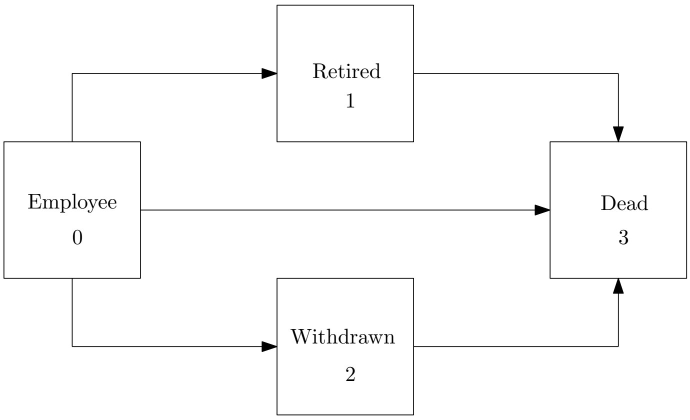
(a) Sample calculation:

$$\begin{gathered}\text{EPV of death benefit for those who retire at age 65} \\ = P(\text{not withdraw from age 50 to 60}) \\ \times P(\text{not die from age 50 to 65}) \\ \times P(\text{not retire at age 60 or 61 or 62 or 63 or 64}) \\ \times P(\text{retire from age 65}) \\ \times v^{15} \\ \times \text{EPV of whole life insurance of 1 on 65 in ILT} \\ = e^{-0.002\times 10}\,{}_{15}p_{50} \times 0.8^5 \times 1 \times v^{15} A_{65} = e^{-0.002\times 10} \times 0.8^5 \times 1 \times {}_{15}E_{50} A_{65}\end{gathered}$$

$$\begin{gathered}\text{EPV of post retirement death benefit} \\ = e^{-0.002\times 10}\left(0.2\left({}_{10}E_{50}A_{60} + 0.8\,{}_{11}E_{50}A_{61} + 0.8^2\,{}_{12}E_{60}A_{62} + 0.8^3\,{}_{13}E_{60}A_{63} + 0.8^4\,{}_{14}E_{60}A_{64}\right) + 0.8^5\,{}_{15}E_{60}A_{65}\right)\end{gathered}$$

Homework 51.1.4

You are given the following withdrawal/retirement model:



You are also given:

(i) The force of mortality depends on the individual's age only and follows the Illustrative Life Table (ILT). That is, $\mu_x^{03} = \mu_x^{13} = \mu_x^{23} = \mu_x$, where μ_x is equal to the force of mortality in the Illustrative Life Table. In addition, deaths are uniformly distributed over each year of age.

(ii)

$$\mu_x^{02} = \begin{cases} 0.001 & \text{if } x < 60 \\ 0.000 & \text{if } x \geq 60 \end{cases}$$

(iii) Employees retire exactly on age 60, 62, 64, or 65.

(iv) 30% of the employees reaching age 60, 62, and 64 retire at that age; 100% of the employees reaching age 65 retire at the age.

(v) Death benefit after retirement: 100,000 at the moment of death.

(vi) $i = 0.06$.

(vii) Selective actuarial values:

k	${}_kE_{55}$	A_{55+k}
5	0.708101	0.36913
6	0.658828	0.38279
7	0.612206	0.39670
8	0.568091	0.41085
9	0.526352	0.42522
10	0.486864	0.43980

(a) Show that the probability that an employee aged x is dead by age $x+t$ is the same as that under the alive-dead model with the transition intensity μ_x.

(b) For an employee currently aged 55, show that the EPV of the post-retirement death benefit is 24,200 to the nearest of 50.

Homework Solution 51.1.5

★★★★★ Difficulty

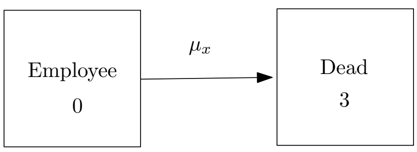
(a)

$$\frac{d}{dt} {}_tp_x^{03} = {}_tp_x^{00} \mu_{x+t}^{03} + {}_tp_x^{01} \mu_{x+t}^{13} + {}_tp_x^{02} \mu_{x+t}^{23}$$

$$= \left({}_tp_x^{00} + {}_tp_x^{01} + {}_tp_x^{02}\right) \mu^{x+t} = \left(1 - {}_tp_x^{03}\right) \mu_{x+t}$$

$$\text{boundary conditions: } {}_0p_x^{00} = 1, \quad {}_0p_x^{03} = 0$$

Now consider the following the alive-death model:



$$\frac{d}{dt}{}_tp_x^{03} = {}_tp_x^{00}\mu_{x+t} = \left(1 - {}_tp_x^{03}\right)\mu_{x+t}$$

$$\text{boundary conditions: } {}_0p_x^{00} = 1, \quad {}_0p_x^{03} = 0$$

Both models have the same derivative $\frac{d}{dt}{}_tp_x^{03}$ and the same boundary conditions. Hence two models generate the same probability ${}_tp_x^{03}$.

(b) Sample calculation:

$$\begin{aligned}
&\text{EPV of death benefit for those who retire at age 65}\\
&= 100{,}000\\
&\times P(\text{not withdraw from age 55 to 60})\\
&\times P(\text{not die from age 55 to 65})\\
&\times P(\text{not retire at age 60, 62, or 64})\\
&\times P(\text{retire from age 65})\\
&\times v^{65-55}\\
&\times \text{EPV of whole life continuous insurance of 1 on 65 in ILT}
\end{aligned}$$

$$= 100{,}000e^{-0.001\times 5}{}_{10}p_{55} \times 0.7^3 \times 1 \times v^{10}\overline{A}_{65} = 100{,}000e^{-0.001\times 5} \times 0.7^3 \times 1 \times {}_{10}E_{55}\overline{A}_{65}$$

$$\text{EPV of post retirement death benefit}$$

$$= \sum_{r=60,62,64,65} \text{EPV if retire at age } r$$

$$= 100{,}000e^{-0.001\times 5}\left(0.3{}_5E_{55}\overline{A}_{60} + 0.7 \times 0.3{}_7E_{55}\overline{A}_{62} + 0.7^2 \times 0.3{}_9E_{55}\overline{A}_{64} + 0.7^3{}_{10}E_{55}\overline{A}_{65}\right)$$

$$= 100{,}000e^{-0.001\times 5} \times \frac{0.06}{\ln 1.06}\left(0.3{}_5E_{55}A_{60} + 0.7 \times 0.3{}_7E_{55}A_{62} + 0.7^2 \times 0.3{}_9E_{55}A_{64} + 0.7^3{}_{10}E_{55}A_{65}\right)$$

$$= 100{,}000e^{-0.001\times 5} \times \frac{0.06}{\ln 1.06}$$

$$\left(0.3 \times 0.708101 \times 0.36913 + 0.7 \times 0.3 \times 0.612206 \times 0.39670 + 0.7^2 \times 0.3 \times 0.526352 \times 0.42522 + 0.7^3 \times 0.486864 \times 0.43980\right)$$

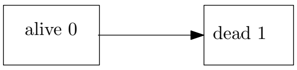
$$= 100{,}000e^{-0.001\times 5} \times \frac{0.06}{\ln 1.06} \times 0.23576 = 24{,}155$$

Chapter 52

Multiple state model: various problems

Example 52.0.1

In the basic survival model, ${}_2p_x = 0.9$. In the alive-dead model, calculate ${}_2p_x^{00}$, ${}_2p_x^{01}$, ${}_2p_x^{10}$, ${}_2p_x^{11}$, ${}_2p_x^{\overline{00}}$, ${}_2p_x^{\overline{11}}$.



Solution 52.0.1

${}_2p_x^{00} = {}_2p_x^{\overline{00}} = {}_2p_x = 0.9$ (once you leave state 1 and can never go back in the alive-death model).

${}_2p_x^{11} = {}_2p_x^{\overline{11}} = 1$ (once dead, always dead).

${}_2p_x^{01} = {}_2q_x = 0.1$, ${}_2p_x^{10} = 0$ (can't change from being dead to alive),

force of transition or transition intensity

If the state variable $Y(t)$ is continuous, then $\mu_x^{ij} = \lim\limits_{h\to 0} \dfrac{{}_hp_x^{ij}}{h}$ for $i \neq j$ is called the force of transition or the transition intensity between state i and state j for age x. This is the counterpart of the force of mortality in the basic alive-dead model and $\mu_x^{01} = \mu_x$.

Example 52.0.2

In the alive(0)-dead(1) model, $\mu_x^{01} = \lim\limits_{h\to 0} \dfrac{{}_hp_x^{01}}{h} = \lim\limits_{h\to 0} \dfrac{{}_hq_x}{h}$. Explain why μ_x^{01} is the force of mortality μ_x.

Solution 52.0.2

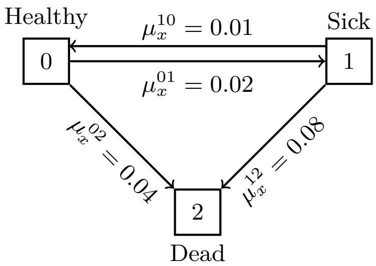
See section 9.2.

Another way to express $\mu_x^{ij} = \lim\limits_{h\to 0} \dfrac{{}_hp_x^{ij}}{h}$ is ${}_hp_x^{ij} = h\mu_x^{ij} + o(h)$, where $o(h)$ is a function that approaches zero faster than h approaches zero. And for a small h, ${}_hp_x^{ij} \approx h\mu_x^{ij}$.

52.1 find probability of being stuck in a state

Example 52.1.1

A 10-year sickness policy is issued to a healthy life age 50. The policy pays a no-claim bonus of 1000 at the end of Year 10 if the insured remains healthy throughout the term of the contract. The transition intensities are constants for all ages. $\delta = 0.06$. Calculate the EPV of the bonus.



Solution 52.1.1

The probability of receiving the bonus is ${}_{10}p_{50}^{\overline{00}}$.

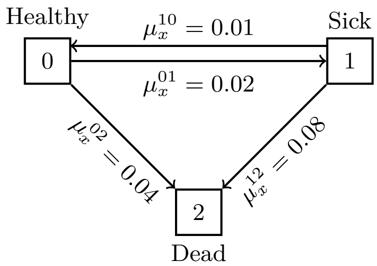
$${}_{10}p_{50}^{\overline{00}} = \exp\left(-\int_0^{10} (\mu_{50+s}^{01} + \mu_{50+s}^{02})ds\right)$$

$$= \exp\left(-\int_0^{10} (0.02 + 0.04)ds\right) = e^{-0.06(10)} = e^{-0.6}$$

EPV: $1000\,{}_{10}p_{50}^{\overline{00}} e^{-10\delta} = 1000e^{-0.12} = 886.92$

Example 52.1.2

The transition intensities are constants for all ages. Calculate the probability that (x) remains sick throughout the next 3 years given that he's sick today.



Solution 52.1.2

$$ {}_3p_x^{\overline{11}} = \exp\left(-\int_0^3 (\mu_{x+s}^{10} + \mu_{x+s}^{12})ds\right)$$

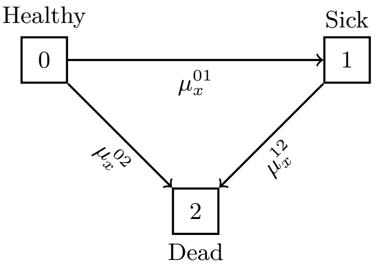
$$= e^{-3(0.09)} = 0.7634$$

52.2 when getting back to a state is the same as being stuck in the state

${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ holds under two situations: (1) you cannot leave a state (such as the death state), and (2) you can leave a state, but if you leave then you can never get back. For any other situation, ${}_tp_x^{ii} > {}_tp_x^{\overline{ii}}$.

Example 52.2.1

For which states does the equation ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ hold?



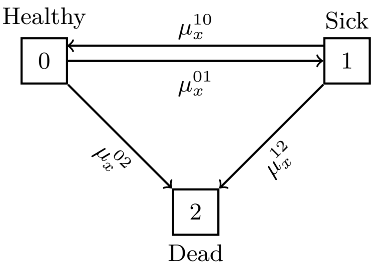
Solution 52.2.1

Clearly, ${}_tp_x^{22} = {}_tp_x^{\overline{22}}$; once dead, always dead.

If you leave state 0, you can't get back. ${}_tp_x^{00} = {}_tp_x^{\overline{00}}$.

Example 52.2.2

For which states does the equation ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ hold?

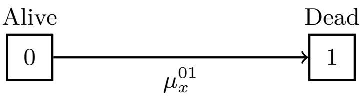


Solution 52.2.2

Only the "dead" state satisfies the equation and ${}_tp_x^{22} = {}_tp_x^{\overline{22}}$. For the other states, ${}_tp_x^{00} > {}_tp_x^{\overline{00}}$ and ${}_tp_x^{11} > {}_tp_x^{\overline{11}}$.

Example 52.2.3

For which states does the equation ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ hold?



Solution 52.2.3

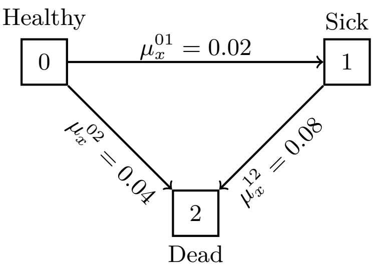
Both the "alive" and the "dead" states satisfy the equation ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}$.

52.3 getting back and being stuck, various transition probabilities

In a word problem, it may not be immediately clear whether you should use ${}_tp_x^{ii}$ or ${}_tp_x^{\overline{ii}}$. If re-entry to state i is impossible, then ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ and it doesn't matter which one you use. However, if re-entry is possible, then ${}_tp_x^{ii} \neq {}_tp_x^{\overline{ii}}$ and water gets muddy. Ask "Is re-entry to state i allowed in the event?" If YES, use ${}_tp_x^{ii}$. If NO, then use ${}_tp_x^{\overline{ii}}$.

Example 52.3.1

A 10-year sickness policy on a healthy life (50) pays 100,000 at the moment when the insured becomes sick. $\delta = 0.06$. Calculate the EPV of this policy.



Solution 52.3.1

The death benefit is paid at t if the insured

- is still in state 0 at t, prob: ${}_tp_{50}^{00}$
- transitions to state 1 during $[t, t+dt]$, prob: $\mu_{50+t}^{01}dt$

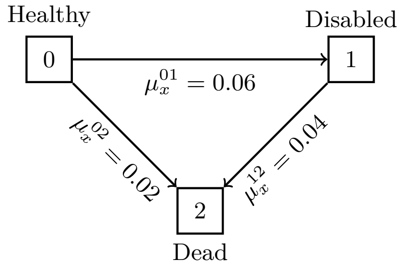
$$100000\int_0^{10} e^{-\delta t}{}_tp_{50}^{00}\mu_{50+t}^{01}dt = 100000\int_0^{10} e^{-\delta t}{}_tp_{50}^{\overline{00}}\mu_{50+t}^{01}dt$$

$$= 100000\int_0^{10} e^{-0.06t}e^{-(0.02+0.04)t}0.02dt$$

$$= \frac{100000\times 2}{12}\left(1 - e^{-0.12\times 10}\right) = 11646.763$$

Example 52.3.2

The transition intensities are constants for all ages. Calculate the probability that a healthy life (x) today is still healthy 10 years from today.

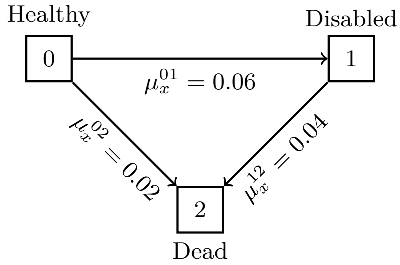


Solution 52.3.2

$$ {}_{10}p_x^{00} = {}_{10}p_x^{\overline{00}} = e^{-(0.06+0.02)10} = e^{-0.8}. $$

Example 52.3.3

The transition intensities are constants for all ages. Calculate the probability that a healthy life (x) today is disabled 10 years from today.



Solution 52.3.3

We want to start from state 0 today and be at state 1 at $t = 10$. We can

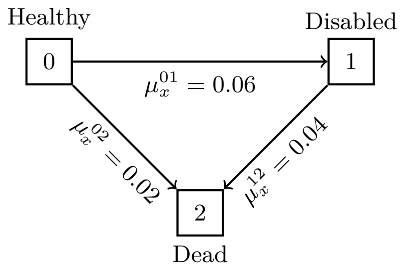
- hang out at state 0 during $[0, t]$, prob: ${}_tp_x^{\overline{00}} = e^{-0.08t}$
- go to state 1 during $[t, t + dt]$, prob: ${}_{dt}p_{x+t}^{01} = \mu_{x+t}^{01}dt + o(dt) \approx 0.06dt$
- hang out in state 1 during $[t + dt, 10] \approx [t, 10]$, prob: ${}_{10-t}p_{x+t}^{\overline{11}} = e^{-0.04(10-t)}$

$$ {}_{10}p_x^{01} = \int_0^{10} {}_tp_x^{\overline{00}}\, \mu_{x+t}^{01}\, {}_{10-t}p_{x+t}^{\overline{11}} dt $$

$$ = \int_0^{10} e^{-0.08t} 0.06 e^{-0.04(10-t)} dt = 0.06e^{-0.4} \int_0^{10} e^{-0.04t} dt = \frac{6e^{-0.4}}{4}\left(1 - e^{-0.4}\right) = 0.3315 $$

Example 52.3.4

The transition intensities are constants for all ages. A healthy insured is age x today. Let A represent the probability that the insured is dead 10 years from today and he's disabled at death. Calculate A.



Solution 52.3.4

We want to complete the path $0 \to 1 \to 2$ in 10 years or less.

$$ A = \int_0^{10} {}_tp_x^{01}\, \mu_{x+t}^{12}\, {}_{10-t}p_{x+t}^{22} dt = \int_0^{10} {}_tp_x^{01}\, \mu_{x+t}^{12} dt $$

From the previous problem,

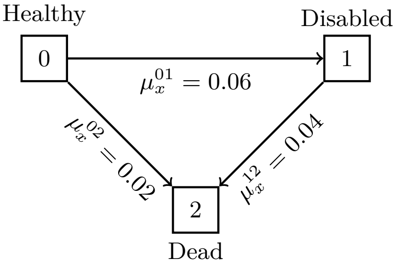
$$ {}_mp_x^{01} = \int_0^{m} {}_tp_x^{\overline{00}}\, \mu_{x+t}^{01}\, {}_{m-t}p_{x+t}^{\overline{11}} dt $$

$$ = \int_0^{m} e^{-0.08t} 0.06 e^{-0.04(m-t)} dt = 0.06e^{-0.04m} \int_0^{m} e^{-0.04t} dt = \frac{0.06}{0.04}\left(e^{-0.04m} - e^{-0.08m}\right) $$

$$ A = \int_0^{10} {}_tp_x^{01}\, \mu_{x+t}^{12} dt = \int_0^{10} \frac{0.06}{0.04}\left(e^{-0.04t} - e^{-0.08t}\right) 0.08 dt = 0.12\left(\frac{1 - e^{-0.4}}{0.04} - \frac{1 - e^{-0.8}}{0.08}\right) = 0.16303 $$

Example 52.3.5

The transition intensities are constants for all ages. A healthy insured is age x today. Let B represent the probability that the insured is dead 10 years from today and he's healthy at death. Calculate B.



Solution 52.3.5

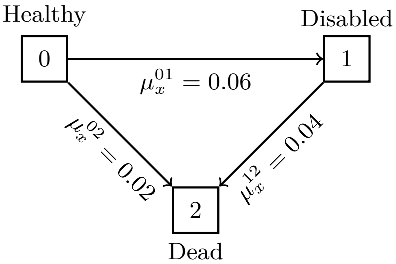
We want to complete the path $0 \to 2$ in 10 years or less.

$$B = \int_0^{10} {}_tp_x^{00} \mu_{x+t}^{02} dt = \int_0^{10} {}_tp_x^{\overline{00}} \mu_{x+t}^{02} dt$$

$$= \int_0^{10} e^{-0.08t} 0.02 dt = 0.02 \frac{1 - e^{-0.8}}{0.08} = 0.1377$$

Example 52.3.6

The transition intensities are constants for all ages. A healthy insured is age x today. Let C represent the probability that the insured is dead 10 years from today.

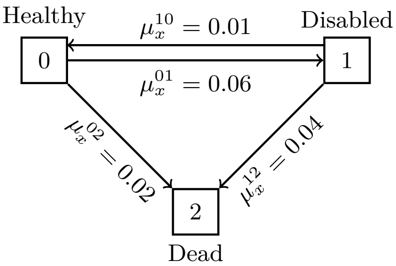


Solution 52.3.6

$$C = A + B = 0.16303 + 0.1377 = 0.30073$$

Example 52.3.7

The transition intensities are constants for all ages. Calculate the probability that a healthy life (x) today is still healthy 10 years from today. Is it ${}_{10}p_x^{00}$ or ${}_{10}p_x^{\overline{00}}$?



Solution 52.3.7

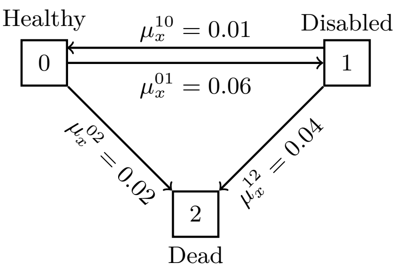
Now ${}_{10}p_x^{00} > {}_{10}p_x^{\overline{00}}$. Which one to use?

When you count the number of people still healthy at $t = 10$, should you include those who return to the healthy state after recovering from prior disabilities? Yes you should. Then the probability is ${}_{10}p_x^{00}$.

There's no closed-form formula for ${}_{10}p_x^{00}$. We can use the Euler method to approximate ${}_{10}p_x^{00}$.

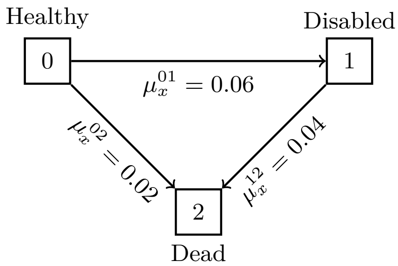
Example 52.3.8

The transition intensities are constants for all ages. Calculate the probability that a healthy life (x) today is ever disabled during the next 10 years.



Solution 52.3.8

The insured can travel back and forth between state 0 and state 1 repeatedly and have many disability relapses. Do disability relapses matter in this problem? Surprisingly NO. The event "ever being disabled" is the same as walking through the path $0 \to 1$ at least once, which is the same as having the 1st period of disability. We can simplify the diagram into:



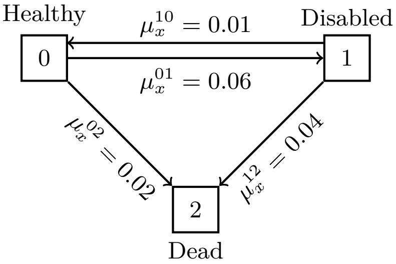
The insured can (1) hang out at state 0 during $[0, t]$, prob: ${}_tp_x^{\overline{00}}$, and (2) move from state 0 to 1 in the next instant, prob: $\mu_{x+t}^{01} dt$. The probability of making these two moves is $p(t) = {}_tp_x^{\overline{00}} \mu_{x+t}^{01} dt$. Next, sum $p(t)$ from $t = 0$ to $t = 10$: $\int_0^{10} {}_tp_x^{\overline{00}} \mu_{x+t}^{01} dt = \int_0^{10} e^{-0.08t} 0.06 dt = \frac{6}{8}\left(1 - e^{-0.8}\right) = 0.4130$.

Example 52.3.9

The transition intensities are constants for all ages. Consider two probabilities:

- A, as calculated in the last problem, is the probability that a healthy life (x) today is disabled at some point during the next 10 years.
- $B = {}_{10}p_x^{01}$ is the probability that a healthy life (x) today is disabled at the end of Year 10.

Actuary Moray is puzzled by the fact that there's an exact solution to A yet we have to use the Euler method to approximate B. Help Moray understand why.



Solution 52.3.9

A is the probability that the *first* disability occurs during the next 10 years.

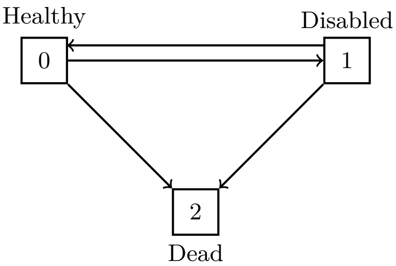
B is the probability that a healthy life now becomes newly disabled at the end of Year 10 and that this disability is the n-th time (where $n = 1, 2, \ldots$) that the insured is disabled during the first 10 years. Clearly, B is much harder to find.

Example 52.3.10

Which expression is the probability that a healthy life (x) today is disabled during the first 10 years and remains disabled throughout the remainder of the first 10 years?

(A) $\int_0^{10} {}_tp_x^{\overline{00}} \mu_{x+t}^{01} {}_{10-t}p_{x+t}^{\overline{11}} dt$

(B) $\int_0^{10} {}_tp_x^{01} {}_{10-t}p_{x+t}^{\overline{11}} dt$



Solution 52.3.10

A is the probability that the insured's *first* disability lasts throughout the remainder of the first 10 years. B is the probability that the insured's *any* disability lasts throughout the remainder of the first 10 years. B is correct.

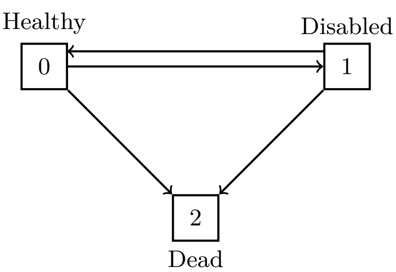
An insured can have many "healthy this month, disabled next month" cycles before finally becoming disabled continuously throughout the remainder of the first 10 years. Such a scenario is discarded in A but is captured in B.

Example 52.3.11

A 10-year disability insurance is issued to a healthy life (x). Premiums are payable continuously at the rate of P per year while the insured is healthy. Which expression is the EPV of the premiums?

(A) $P \int_0^{10} e^{-\delta t} {}_tp_x^{\overline{00}} dt$

(B) $P \int_0^{10} e^{-\delta t} {}_tp_x^{00} dt$

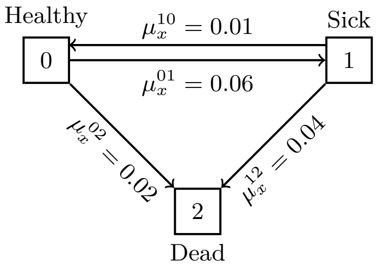


Solution 52.3.11

If the insured returns to the healthy state after recovering from disability during the term of the policy, will he pay the premium? YES. The EPV is A.

Example 52.3.12

A 10-year sickness insurance policy on a healthy life (x) pays a first-sickness-recovery bonus. If the insured is sick but later recovers from his first sickness during the term of the contract, a bonus of 100 is immediately paid upon recovery. The transition intensities are constants for all ages. $\delta = 0.07$. Calculate the EPV of the bonus.



Solution 52.3.12

The bonus is paid if the insured walks through the path $0 \to 1 \to 0$ in 10 years. The insured needs to do the following:

- stay in state 0 during $[0, t]$, prob: ${}_tp_x^{\overline{00}}$,
- transition to state 1 during $[t, t + dt]$, prob: $\mu_{x+t}^{01} dt$,
- stay in state 1 during $[t, t + u]$, prob: ${}_up_{x+t}^{\overline{11}}$, and finally
- transition to state 0 during $[t + u, t + u + du]$, prob: $\mu_{x+t+u}^{10} du$

After making the above moves, the insured will get the bonus is at $t + u$. The constraints are

$$t \geq 0, \quad u \geq 0, \quad 0 \leq t + u \leq 10$$

$$100 \int_{t=0}^{10} \int_{u=0}^{10-t} e^{-\delta(t+u)} {}_tp_x^{\overline{00}} \mu_{x+t}^{01} {}_up_{x+t}^{\overline{11}} \mu_{x+t+u}^{10} du dt$$

$$= 100 \int_{t=0}^{10} \int_{u=0}^{10-t} e^{-0.07(t+u)} e^{-0.08t} 0.06 e^{-0.05u} 0.01 du dt$$

$$= 0.06 \int_0^{10} e^{-0.15t} \int_0^{10-t} e^{-0.12u} du dt = 0.06 \int_0^{10} \frac{1 - e^{-0.12(10-t)}}{0.12} e^{-0.15t} dt$$

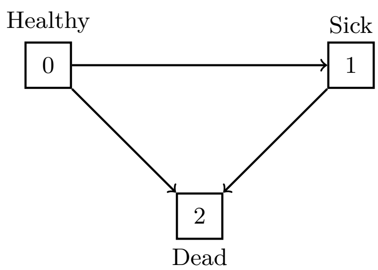
$$= 0.5 \left(\int_0^{10} e^{-0.15t} dt - e^{-1.2} \int_0^{10} e^{-0.03t} dt \right)$$

$$= 0.5 \left(\frac{1 - e^{-1.5}}{0.15} - e^{-1.2} \frac{1 - e^{-0.3}}{0.03} \right) = 1.2885$$

52.4 Check your knowledge

Homework 52.4.1

(MLC: Spring 2016 Q4) A 5-year sickness insurance policy is based on the following Markov model:



You are given the following constant forces of transition:

(i) $\mu^{01} = 0.05$

(ii) $\mu^{10} = 0.02$

(iii) $\mu^{02} = 0.01$

(iv) $\mu^{12} = 0.06$

Calculate the probability that a Healthy life will become Sick exactly once during the 5 years and remain continuously Sick from that point until the end of the 5 years.

Homework Solution 52.4.1

★★★★☆ Difficulty

$$\int_0^5 {}_tp_x^{\overline{00}}\mu_{x+t}^{01}{}_{5-t}p_{x+t}^{\overline{11}}dt = \int_0^5 e^{-(0.05+0.01)t}0.05e^{-(5-t)(0.02+0.06)}dt = 0.17624544$$

Homework 52.4.2

(spring 2012 MLC Q12) Employees in Company ABC can be in:

State 0: Non-executive employee

State 1: Executive employee

State 2: Terminated from employment

John joins Company ABC as a non-executive employee at age 30. You are given:

(i) $\mu^{01} = 0.01$ for all years of service

(ii) $\mu^{02} = 0.06$ for all years of service

(iii) $\mu^{12} = 0.02$ for all years of service

(iv) Executive employees never return to the non-executive employee state.

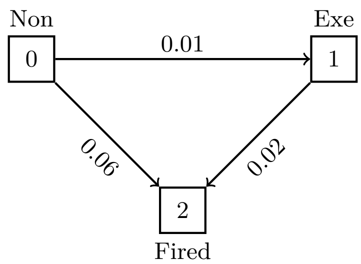
(v) Employees terminated from employment never get rehired.

(vi) The probability that John lives to age 65 is 0.9, regardless of state.

Calculate the probability that John will be an executive employee of Company ABC at age 65.

Homework Solution 52.4.2

★★★★☆ Difficulty



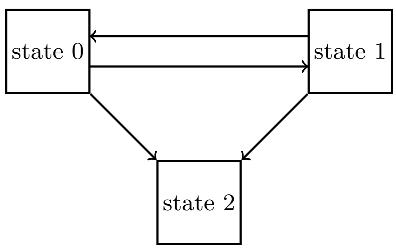
The probability that John will be an executive employee of Company ABC at age 65 is:

$${}_{35}p_{30} \times {}_{35}p_{30}^{01} = 0.9{}_{35}p_{30}^{01}{}_{35}p_{30}^{01} = \int_0^{35} {}_tp_{30}^{\overline{00}}\mu_{30+t}^{01}{}_{35-t}p_{30+t}^{\overline{11}}dt = \int_0^{35} e^{-(0.01+0.06)t}0.01e^{-0.02(35-t)}dt = 0.258$$

$$0.9(0.258) = 0.2322$$

Homework 52.4.3

(MLC: Spring 2014 Q3) A continuous Markov process is modeled by the following multiple state diagram:



You are given the following constant transition intensities:

(i) $\mu^{01} = 0.08$

(ii) $\mu^{02} = 0.04$

(iii) $\mu^{10} = 0.10$

(iv) $\mu^{12} = 0.05$

For a person in State 1, calculate the probability that the person will continuously remain in State 1 for the next 15 years.

Homework Solution 52.4.3

★★★☆☆ Difficulty

$$
{}_{15}p_x^{\overline{11}} = \exp\left(-\int_0^{15}(\mu_{x+s}^{10} + \mu_{x+s}^{12})ds\right) = e^{-15(0.15)} = 0.1054
$$

Homework 52.4.4

(Exam MLC: Spring 2012 Q28) You are using Euler's method to calculate estimates of probabilities for a multiple state model with states $0, 1, 2$. You are given:

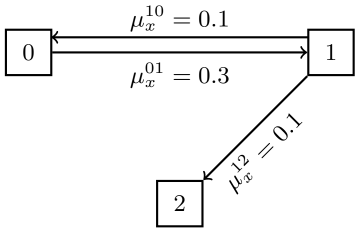
(i) The only possible transitions between states are: 0 to 1, 1 to 0, and 1 to 2

(ii) For all x, $\mu_x^{01} = 0.3$, $\mu_x^{10} = 0.1$, $\mu_x^{12} = 0.1$

(iii) Your step size is 0.1.

(iv) You have calculated that ${}_{0.6}p_x^{00} = 0.8370$, ${}_{0.6}p_x^{01} = 0.1588$, ${}_{0.6}p_x^{02} = 0.0042$,

Calculate the estimate of ${}_{0.8}p_x^{01}$ using the specified procedure.



Homework Solution 52.4.4

★★★☆☆ Difficulty

$$
\frac{d}{dt}{}_tp_x^{00} = t\left[{}_tp_x^{01}\mu_{x+t}^{10} - {}_tp_x^{00}\mu_{x+t}^{01}\right] = t\left[0.1{}_tp_x^{01} - 0.3{}_tp_x^{00}\right]
$$

$$
\frac{d}{dt}{}_tp_x^{01} = t\left[{}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12})\right] = t\left[0.3{}_tp_x^{00} - 0.2{}_tp_x^{01}\right]
$$

$$
{}_{0.7}p_x^{00} \approx {}_{0.6}p_x^{00} + 0.1(0.1{}_{0.6}p_x^{01} - 0.3{}_{0.6}p_x^{00}) = 0.8370 + 0.1(0.1(0.1588) - 0.3(0.8370)) = 0.813478
$$

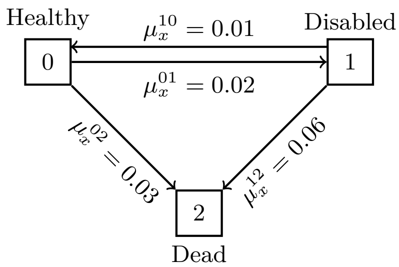
$$
{}_{0.7}p_x^{01} \approx {}_{0.6}p_x^{01} + 0.1(0.3{}_{0.6}p_x^{00} - 0.2{}_{0.6}p_x^{01}) = 0.1588 + 0.1(0.3(0.8370) - 0.2(0.1588)) = 0.180734
$$

$$
{}_{0.8}p_x^{01} \approx {}_{0.7}p_x^{01} + 0.1(0.3{}_{0.7}p_x^{00} - 0.2{}_{0.7}p_x^{01}) = 0.180734 + 0.1(0.3(0.813478) - 0.2(0.180734)) = 0.20152366
$$

Homework 52.4.5

A disability insurance is issued to healthy life (x). The transition intensities are constants for all ages. Let the contract issue time be time zero. Use the Euler method. Set $h = 1/12$.

- ${}_{3h}p_x^{00}$, ${}_{3h}p_x^{01}$, ${}_{3h}p_x^{11}$, ${}_{3h}p_x^{10}$
- estimate ${}_{3h}p_x^{00} - {}_{3h}p_x^{\overline{00}}$



Homework Solution 52.4.5

★★★★☆ Difficulty

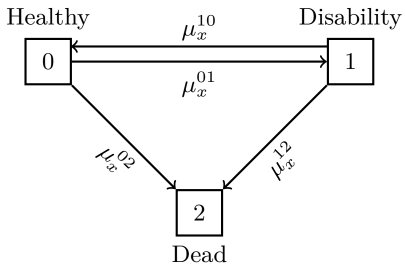
t	${}_tp_x^{00}$	${}_tp_x^{01}$	${}_tp_x^{02}$
0	1.00000	0.00000	0.00000
h	0.99583	0.00167	0.00250
$2h$	0.99169	0.00332	0.00500
$3h$	0.98756	0.00495	0.00749

$$_{3h}p_x^{\overline{00}} = \int_0^{3/12} e^{-0.05t} dt = \frac{1 - e^{-0.05\times 3/12}}{0.05} = 0.24844$$

$$_{3h}p_x^{00} - {}_{3h}p_x^{\overline{00}} = 0.98756 - 0.24844 = 0.73912$$

Homework 52.4.6

A 10-year disability insurance policy is issued to healthy life (x). The policy pays 100,000 immediately at the onset of disability.



Which expression is the EPV for this policy?

(A) $100000 \int_0^{10} e^{-\delta t} {}_tp_x^{00} \mu_{x+t}^{01} dt$

(B) $100000 \int_0^{10} e^{-\delta t} {}_tp_x^{\overline{00}} \mu_{x+t}^{01} dt$

Homework Solution 52.4.6

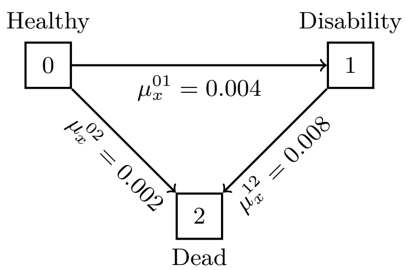
★★★★☆ Difficulty

A is correct. The insured can be newly disabled many times during the term of the contract and each new disability triggers the benefit. A counts for this while B is the EPV of the benefit for the first disability during the term of the contract. There's no exact way to calculate the integral $100000 \int_0^{10} e^{-\delta t} {}_tp_x^{00} \mu_{x+t}^{01} dt$ as there's no exact way to find ${}_tp_x^{00}$.

Homework 52.4.7

A combined 10-year term life and disability insurance policy is issued to healthy life (x). The policy pays 100,000 immediately on death or the onset of disability. No further benefit is paid in the event of death after a prior disability claim has been paid.

The transition intensities are constants for all ages. $\delta = 0.05$. Calculate the EPV of this policy.



Homework Solution 52.4.7

★★★★☆ Difficulty

We can ignore path $1 \to 2$ because death after disability will not trigger the death benefit.

$$100000 \int_0^{10} e^{-\delta t} {}_tp_x^{00} (\mu_{x+t}^{01} + \mu_{x+t}^{02}) dt = 100000 \int_0^{10} e^{-\delta t} {}_tp_x^{\overline{00}} (\mu_{x+t}^{01} + \mu_{x+t}^{02}) dt$$

$$= 100000 \int_0^{10} e^{-0.05t} e^{-0.006t} 0.006 dt = 100000 \int_0^{10} e^{-0.056t} 0.006 dt = 100000 \times \frac{6}{56} \left(1 - e^{-0.056\times 10}\right) = 4594.1886$$

Homework 52.4.8

Same as the last problem EXCEPT that the limitation "no further benefit is paid in the event of death after a prior disability claim has been paid" is removed. Calculate the EPV of this policy.

Homework Solution 52.4.8

★★★★★ Difficulty

METHOD 1. EPV of the death benefit if the insured is healthy at death:

$$100000\int_0^{10} e^{-\delta t}\,{}_tp_x^{00}\mu_{x+t}^{02}dt = 100000\int_0^{10} e^{-0.05t}e^{-0.006t}0.002dt = 100000\times\frac{2}{56}\left(1-e^{-0.56}\right) = 1531.3962$$

EPV of the disability benefit if the insured is continuously disabled throughout the remainder of the 10-year term:

$$100000\int_0^{10} e^{-\delta t}\,{}_tp_x^{00}\mu_{x+t}^{01}\,{}_{10-t}p_{x+t}^{\overline{11}}dt = 100000\int_0^{10} e^{-0.05t}e^{-0.006t}0.004e^{-(10-t)0.008}dt$$

$$= 100000(0.004)e^{-0.08}\int_0^{10} e^{-0.048t}dt = 100000(0.004)e^{-0.08}\frac{1-e^{-0.48}}{0.048} = 2932.5607$$

The insured will get paid twice if he walks through the path $0\to1\to2$ during the 10-year term: (1) the disability benefit 100,000 at t, and (2) the death benefit 100,000 at $t+u$ if he makes the following moves:

(a) is continuously healthy during $[0,t]$; prob: ${}_tp_x^{\overline{00}} = e^{-0.006t}$

(b) becomes disabled during $[t,t+dt]$; prob: $\mu_{x+t}^{01}dt = 0.004dt$

(c) is continuously disabled during $[t+dt,t+dt+u]\approx[t,t+u]$; prob: ${}_up_{x+t}^{\overline{11}} = e^{-0.008u}$

(d) moves to state 2 during $[t+u,t+u+du]$, prob: $\mu_{x+t+u}^{12}du = 0.008du$.

For each (t,u) pair where $0<t+u<10$, the EPV of the double payments is

$$g(t,u) = 100,000(e^{-\delta t}+e^{-\delta(t+u)})e^{-0.006t}0.004e^{-0.008u}0.008dudt$$

$$\int_{t=0}^{10}\int_{u=0}^{10-t} g(t,u) = 100,000\int_{t=0}^{10}\int_{u=0}^{10-t}(e^{-0.05t}+e^{-0.05(t+u)})e^{-0.006t}0.004e^{-0.008u}0.008dudt$$

$$= 100000(0.004)(0.008)\int_0^{10} e^{-0.056t}\left(\int_0^{10-t}(1+e^{-0.05u})e^{-0.008u}du\right)dt = 240.6666$$

$$\textit{Total EPV}:\quad 1531.3962+2932.5607+240.6666 = 4704.6235$$

If double payments are not allowed, the EPV for death after disability is

$$100,000\int_{t=0}^{10}\int_{u=0}^{10-t}(e^{-0.05t}+0)e^{-0.006t}0.004e^{-0.008u}0.008dudt = 130.23171$$

$$\textit{Total EPV}:\quad 1531.3962+2932.5607+130.23171 = 4594.1886$$

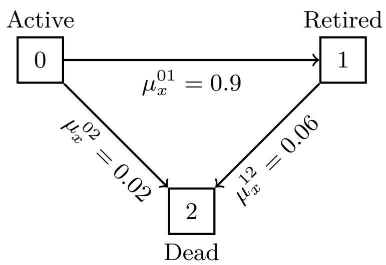
METHOD 2. EPV of the death benefit if the insured is disabled at death:

$$100000\int_0^{10}\int_0^{10-t} e^{-\delta(t+u)}\,{}_tp_x^{00}\mu_{x+t}^{01}\,{}_up_{x+t}^{\overline{11}}\mu_{x+t+u}^{12}dudt = 100000\int_0^{10}\int_0^{10-t} e^{-0.05(t+u)}e^{-0.006t}0.004e^{-0.008u}0.008dudt = 110.4349$$

$$\textit{Total EPV}:\quad 4594.1886+110.4349 = 4704.6235$$

Homework 52.4.9

A pension plan provides a benefit of 100,000 payable on death regardless of whether death occurs before or after retirement. The transition intensities are constants for all ages. $\delta = 0.05$. Calculate the EPV of this policy for an active member currently age x.



Homework Solution 52.4.9

★★★★☆ Difficulty
EPV for the path $0 \to 2$:

$$100000 \int_0^\infty e^{-\delta t}\,{}_tp_x^{00}\mu_{x+t}^{02}dt = 100000 \int_0^\infty e^{-0.05t}e^{-0.92t}0.02dt = 2061.8557$$

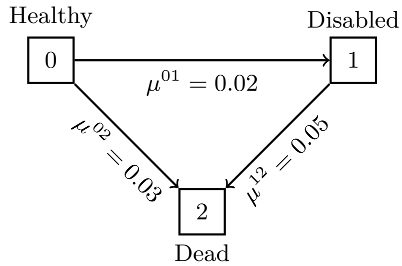
EPV for the path $0 \to 1 \to 2$ (the death benefit is paid at age $x+t+u$):

$$\begin{aligned}
&100000 \int_0^\infty {}_tp_x^{\overline{00}}\mu_{x+t}^{01} \int_0^\infty e^{-\delta(t+u)}\,{}_up_{x+t}^{\overline{11}}\mu_{x+t+u}^{12}dudt \\
&= 100000 \int_0^\infty e^{-0.92t}0.9 \int_0^\infty e^{-0.05(t+u)}e^{-0.06u}0.06dudt \\
&= 100000 \times 0.9 \times 0.06 \int_0^\infty e^{-0.97t}dt \int_0^\infty e^{-0.11u}du \\
&= \frac{100000 \times 0.9 \times 0.06}{0.97 \times 0.11} = 50609.185
\end{aligned}$$

$$\text{Total:} \quad 2061.8557 + 50609.185 = 52671.041$$

Homework 52.4.10

(Exam MLC: Fall 2013 Q10) Calculate the conditional probability that a Healthy life on January 1, 2004 is still Healthy on January 1, 2014, given that this person is not Dead on January 1, 2014.



Homework Solution 52.4.10

★★★☆☆ Difficulty
A =healthy today and healthy 10 years from today. B =healthy today, disabled at some point during the next 10 years and remain disabled during the reminder of the 10 year period.

$$P(A) = {}_{10}p_x^{00} = {}_{10}p_x^{\overline{00}} = e^{-(\mu^{01}+\mu^{02})10} = e^{-0.05\times 10} = e^{-0.5}$$

$$P(B) = \int_0^{10} {}_tp_x^{\overline{00}}\mu^{01}\,{}_{10-t}p_{x+t}^{\overline{11}}dt$$

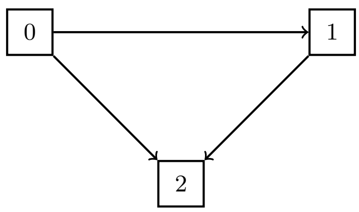
$$= \int_0^{10} e^{-(\mu^{01}+\mu^{02})t}\mu^{01}e^{-\mu^{12}(10-t)}dt = \int_0^{10} e^{-0.05t}0.02e^{-0.05(10-t)}dt = 0.2e^{-0.5}$$

$$P(A \mid A \bigcup B) = \frac{P(A)}{P(A)+P(B)} = \frac{1}{1+0.2} = 0.8333$$

Homework 52.4.11

(Exam MLC: Fall 2013 Q21) You are pricing an automobile insurance on (x). The insurance pays 10,000 immediately if (x) gets into an accident within 5 years of issue. The policy pays only for the first accident and has no other benefits.

(i) You model (x)'s driving status as a multi-state model with the following 3 states:



0 - low risk, without an accident
1 - high risk, without an accident
2 - has had an accident

(ii) (x) is initially in state 0.

(iii) The following transition intensities for $0 \leq t \leq 5$

$$\mu_{x+t}^{01} = 0.20 + 0.10t$$
$$\mu_{x+t}^{02} = 0.05 + 0.05t$$
$$\mu_{x+t}^{12} = 0.15 + 0.01t^2$$

(iv) ${}_3p_x^{01} = 0.4174$

(v) $\delta = 0.02$

(vi) The continuous function $g(t)$ is such that the expected present value of the benefit up to time a equals $\int_0^a g(t)dt$, $0 \leq a \leq 5$, where t is the time of the first accident.

Calculate $g(3)$.

Homework Solution 52.4.11

★★★★★ Difficulty

The most difficult task is to figure out what $g(t)$ means. $g(t)$ is the EPV of the single claim at t. A claim occurs when state 0 or 1 moves to state 2.

$$g(t) = 10000e^{-\delta t}({}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12})$$

$$g(3) = 10000e^{-3\delta}({}_3p_x^{00}\mu_{x+3}^{02} + {}_3p_x^{01}\mu_{x+3}^{12})$$

You are already given ${}_3p_x^{01} = 0.4174$

$${}_3p_x^{00} = {}_3p_x^{\overline{00}} = \exp\left(-\int_0^3 (\mu_{x+t}^{01} + \mu_{x+t}^{02})dt\right)$$

$$= \exp\left(-\int_0^3 (0.20 + 0.10t + 0.05 + 0.05t)dt\right) = e^{-1.425}$$

$${}_3p_x^{00}\mu_{x+3}^{02} + {}_3p_x^{01}\mu_{x+3}^{12} = e^{-1.425}(0.05 + 0.05 \times 3) + 0.4174(0.15 + 0.01 \times 3^2) = 0.148276$$

$$g(3) = 10000e^{-0.02\times 3}0.148276 = 1396.4108$$

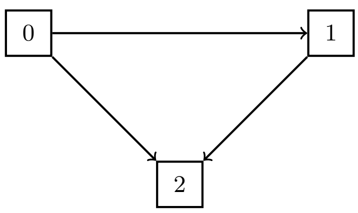
Homework 52.4.12

(Exam MLC: Fall 2012 Q12) A party of scientists arrives at a remote island. Unknown to them, a hungry tyrannosaur lives on the island. You model the future lifetimes of the scientists as a three-state model, where:

0 - State 0: no scientists have been eaten.

1 - State 1: exactly one scientist has been eaten.

2 - State 2: at least two scientists have been eaten.



You are given:

(i) Until a scientist is eaten, they suspect nothing, so $\mu_t^{01} = 0.01 + 0.02 \times 2^t$, $t > 0$

(ii) Until a scientist is eaten, they suspect nothing, so the tyrannosaur may come across two together and eat both, with $\mu_t^{02} = 0.5\mu_t^{01}$, $t > 0$

(iii) After the first death, scientists become much more careful, so $\mu_t^{12} = 0.01$, $t > 0$

Calculate the probability that no scientists are eaten in the first year.

Homework Solution 52.4.12

★★★★☆ Difficulty

$$ {}_1p_x^{00} = {}_1p_x^{\overline{00}} = \exp\left(-\int_0^1 (\mu_{x+t}^{01} + \mu_{x+t}^{02})dt\right) = \exp\left(-\int_0^1 1.5\mu_{x+t}^{01}dt\right) $$

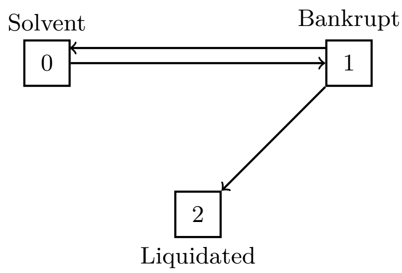
$$ \int_0^1 1.5\mu_{x+t}^{01}dt = \int_0^1 1.5(0.01 + 0.02 \times 2^t)dt $$

$$ = 1.5\Big(0.01t + \frac{0.02 \times 2^t}{\ln 2}\Big)_0^1 = 1.5\Big(0.01 + \frac{0.02}{\ln 2}\Big) = 0.05828 $$

$$ {}_1p_x^{00} = e^{-0.05828} = 0.9434 $$

Homework 52.4.13

(Exam MLC: Fall 2012 Q16) You are evaluating the financial strength of companies based on the following multiple state model:



For each company, you assume the following constant transition intensities:

(i) $\mu^{01} = 0.02$

(ii) $\mu^{10} = 0.06$

(iii) $\mu^{12} = 0.10$

Using Kolmogorov's forward equation with step $h = 1/2$, calculate the probability that a company currently Bankrupt will be Solvent at the end of one year.

Homework Solution 52.4.13

★★★★☆ Difficulty

$$ \frac{d}{dt}{}_tp_x^{10} = {}_tp_x^{11}\mu_{x+t}^{10} - {}_tp_x^{10}\mu_{x+t}^{01} = 0.06{}_tp_x^{11} - 0.02{}_tp_x^{10} $$

,

$$ \frac{d}{dt}{}_tp_x^{10}\Big|_{t=0} = 0.06(1) - 0.02(0) = 0.06 $$

,

$$ \frac{d}{dt}{}_tp_x^{11} = {}_tp_x^{10}\mu_{x+t}^{01} - {}_tp_x^{11}(\mu_{x+t}^{10} + \mu_{x+t}^{12}) = 0.02{}_tp_x^{10} - 0.16{}_tp_x^{11} $$

,

$$ \frac{d}{dt}{}_tp_x^{11}\Big|_{t=0} = 0.2(0) - 0.16(1) = -0.16 $$

,

$$ {}_hp_x^{10} \approx {}_0p_x^{10} + h\frac{d}{dt}{}_tp_x^{10}\Big|_{t=0} = 0 + 0.06/2 = 0.03 $$

$$ {}_hp_x^{11} \approx {}_0p_x^{11} + h\frac{d}{dt}{}_tp_x^{11}\Big|_{t=0} = 1 - 0.16/2 = 0.92 $$

$$ \frac{d}{dt}{}_tp_x^{10}\Big|_{t=h} = 0.06(0.92) - 0.02(0.03) = 0.0546 $$

,

$$ {}_{2h}p_x^{10} \approx {}_hp_x^{10} + h\frac{d}{dt}{}_tp_x^{10}\Big|_{t=h} = 0.03 + 0.0546/2 = 0.0573 $$

Homework 52.4.14

The mortality is $S(x) = 1 - \frac{x}{100}$, $0 \le x \le 100$. Use Kolmogorov's forward equation and the step size $h = \frac{1}{12}$. Calculate the probability that age 40 survives at least two months.

Homework Solution 52.4.14

★★★★☆ Difficulty

For a simple alive-dead model, the KM forward equation is ${}_{t+h}p_x \approx {}_tp_x(1 - h\mu_{x+t})$. Set $h = \frac{1}{12}$ and $t = 0$. Notice that $\mu_x = \dfrac{1}{100 - x}$ and ${}_0p_x = 1$.

$$ {}_{\frac{1}{12}}p_{40} \approx {}_0p_{40}(1 - h\mu_{40}) = 1\left(1 - \frac{1}{12} \times \frac{1}{100 - 40}\right) = 0.998611 $$

$$ {}_{\frac{2}{12}}p_{40} \approx {}_{\frac{1}{12}}p_{40}(1 - h\mu_{40+\frac{1}{12}}) $$

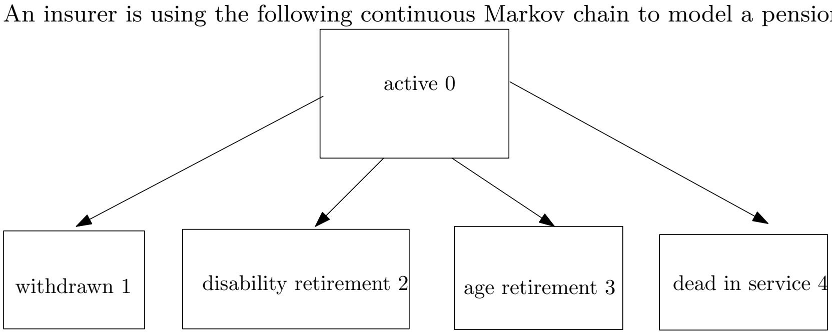
$$ = 0.998611\left(1 - \frac{1}{12} \times \frac{1}{100 - 40 - \frac{1}{12}}\right) = 0.997222 $$

The true value is

$$ \frac{S(40 + \frac{2}{12})}{S(40)} = \frac{1 - \dfrac{40 + \frac{2}{12}}{100}}{1 - \dfrac{40}{100}} = \frac{60 - \frac{2}{12}}{60} = 0.997222 $$

Homework 52.4.15

An insurer is using the following continuous Markov chain to model a pension plan:



(a) Derive the formula ${}_tp_x^{00} = \exp\left(-\int_0^t \sum_{j=1}^{4} \mu_{x+s}^{0j} ds\right)$

(b) State the major assumptions made in the above formula.

(c) Derive the Kolmogorov's forward equations for ${}_tp_x^{01}$. Show that the solution to the Kolmogorov's forward equation for ${}_tp_x^{01}$ is ${}_tp_x^{01} = \int_0^t {}_sp_x^{00}\mu_{x+s}^{01} ds$.

Homework Solution 52.4.15

★★★★☆ Difficulty

(*a*)

$$ {}_{t+h}p_x^{00} = {}_tp_x^{00}\ {}_hp_{x+t}^{00} $$

For a small $h > 0$:

$$ {}_hp_{x+h}^{00} = 1 - \sum_{j=1}^{4} {}_hp_{x+h}^{0j} = 1 - \sum_{j=1}^{4}\left(\mu_{x+h}^{0j} + o(h)\right) $$

$$ \Rightarrow {}_{t+h}p_x^{00} = {}_tp_x^{00} - {}_tp_x^{00}\, h\sum_{j=1}^{4}\left(\mu_{x+h}^{0j} + o(h)\right) $$

$$ \frac{{}_{t+h}p_x^{00} - {}_tp_x^{00}}{h} = -{}_tp_x^{00}\sum_{j=1}^{4}\mu_{x+h}^{0j} + \frac{o(h)}{h} $$

$$ \lim_{h \to 0+} \frac{{}_{t+h}p_x^{00} - {}_tp_x^{00}}{h} = -{}_tp_x^{00}\sum_{j=1}^{4}\mu_x^{0j} + 0 $$

$$ \frac{d}{dt}{}_tp_x^{00} = -{}_tp_x^{00}\sum_{j=1}^{4}\mu_x^{0j} $$

$$ \frac{d}{dt}\ln {}_tp_x^{00} = -\sum_{j=1}^{4}\mu_x^{0j} $$

Integrate both sides and use the boundary condition ${}_0p_x^{00} = 1$:

$$ {}_tp_x^{00} = \exp\left(- \int_0^t \sum_{j=1}^{4} \mu_{x+s}^{0j} ds \right) $$

(b) Major assumptions:

- transition probabilities depend only on the current age and the current state (Markov property)
- P(transitions in a short intervals $h = 0$)

(c)

$$ {}_{t+h}p_x^{01} = {}_tp_x^{01}\, {}_hp_{x+t}^{11} + {}_tp_x^{00}\, {}_hp_{x+t}^{01} = {}_tp_x^{01} + {}_tp_x^{00}\, {}_hp_{x+t}^{01} \text{ because } {}_hp_{x+t}^{11} = 1 $$

$$ \mu_x^{01} = \lim_{h \to 0+} \frac{{}_hp_x^{01}}{h} \text{ definition} \qquad \to {}_hp_x^{01} = h\, \mu_x^{01} + o(h), \quad {}_hp_{x+t}^{01} = h\, \mu_{x+t}^{01} + o(h) $$

$$ {}_{t+h}p_x^{01} - {}_tp_x^{01} = h\, {}_tp_x^{00}\, \mu_{x+t}^{01} + o(h) $$

$$ \frac{d}{dt} {}_tp_x^{01} = \lim_{h \to 0+} \frac{{}_{t+h}p_x^{01} - {}_tp_x^{01}}{h} = {}_tp_x^{00}\, \mu_{x+t}^{01} $$

The above formula is the Kolmogorov's forward equations for ${}_tp_x^{01}$. Integrate both sides of the above formula from $t = 0$ to t:

$$ {}_tp_x^{01} - {}_0p_x^{01} = \int_0^t {}_sp_x^{00}\, \mu_{x+s} ds $$

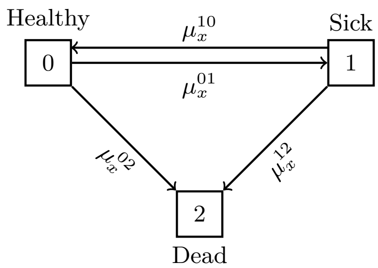
$$ {}_0p_x^{01} = 0, \quad \Rightarrow {}_tp_x^{01} = \int_0^t {}_sp_x^{00}\, \mu_{x+s} ds $$

Homework 52.4.16

(4 points) For the following Markov model, derive the following Kolmogorov's forward equations:

(a) $\dfrac{d}{dt} {}_tp_x^{02} = {}_tp_x^{00}\, \mu_{x+t}^{02} + {}_tp_x^{01}\, \mu_{x+t}^{12}$

(b) $\dfrac{d}{dt} {}_tp_x^{01} = {}_tp_x^{00}\, \mu_{x+t}^{01} - {}_tp_x^{01} \left(\mu_{x+t}^{10} + \mu_{x+t}^{12} \right)$



Homework Solution 52.4.16

★★★★☆ Difficulty

(a)

$$ {}_{t+h}p_x^{02} = {}_tp_x^{00}\, {}_hp_{x+t}^{02} + {}_tp_x^{01}\, {}_hp_{x+t}^{12} + {}_tp_x^{02}\, {}_hp_{x+t}^{22} $$

${}_hp_{x+t}^{22} = 0$. For a small $h > 0$:

$$ {}_hp_{x+t}^{02} = h\mu_{x+t}^{02} + o(h), \quad {}_hp_{x+t}^{12} = h\mu_{x+t}^{12} + o(t) $$

$$ {}_{t+h}p_x^{02} = {}_tp_x^{00}\, h\mu_{x+t}^{02} + {}_tp_x^{01}\, h\mu_{x+t}^{12} + {}_tp_x^{02} + o(h) $$

$$ \frac{{}_{t+h}p_x^{02} - {}_tp_x^{02}}{h} {}_{t+h}p_x^{02} = {}_tp_x^{00}\, \mu_{x+t}^{02} + {}_tp_x^{01}\, \mu_{x+t}^{12} + \frac{o(h)}{h} $$

$$ \lim_{h \to 0+} \frac{{}_{t+h}p_x^{02} - {}_tp_x^{02}}{h} = {}_tp_x^{00}\, \mu_{x+t}^{02} + {}_tp_x^{01}\, \mu_{x+t}^{12} + 0 $$

$$ \Rightarrow \frac{d}{dt} {}_tp_x^{02} = {}_tp_x^{00}\, \mu_{x+t}^{02} + {}_tp_x^{01}\, \mu_{x+t}^{12} $$

(b)

$$ {}_{t+h}p_x^{01} = {}_tp_x^{00}\, {}_hp_{x+t}^{01} + {}_tp_x^{01}\, {}_hp_{x+t}^{11} $$

$$ {}_hp_{x+t}^{01} = h\mu_{x+t}^{01} + o(h) $$

$$ {}_hp_{x+t}^{11} = 1 - \left({}_hp_{x+t}^{10} + {}_hp_{x+t}^{12} \right) = 1 - h\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} \right) + o(h) $$

$$
{}_{t+h}p_x^{01} = {}_tp_x^{00}\, h\mu_{x+t}^{01} + {}_tp_x^{01} - {}_tp_x^{01}\, h\, h\left(\mu_{x+t}^{10} + \mu_{x+t}^{12}\right) + o(h)
$$

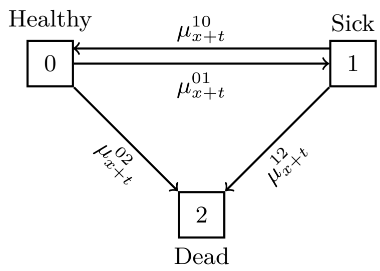
$$
\frac{{}_{t+h}p_x^{01} - {}_tp_x^{01}}{h} = {}_tp_x^{00}\, \mu_{x+t}^{01} - {}_tp_x^{01}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12}\right) + \frac{o(h)}{h}
$$

Take derivative on both sides and you'll get the desired result.

Homework 52.4.17

An insurer issues a 5-year combined death and sickness policy to a healthy life aged 40. You are given:

- $\delta = 0.06$
- The annual rate of premium, P, is payable continuously while the insured is healthy
- Death benefit: 10,000 at the moment of death, with additional 5,000 if the insured is sick at death
- Sickness benefit: 5,000 payable continuously while the insured is sick



Write down the formula for ${}_kV^{(0)}$ and for ${}_kV^{(1)}$.

Homework Solution 52.4.17

★★★★☆ Difficulty

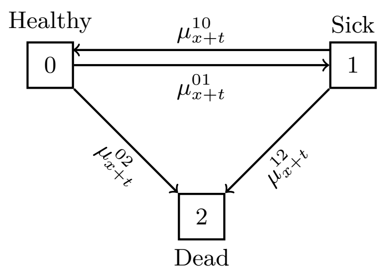
$$
{}_kV^{(0)} = 10{,}000\int_0^{5-k} e^{-\delta t}\,{}_tp_{40+k}^{00}\mu_{40+k+t}^{02}dt + 15{,}000\int_0^{5-k} e^{-\delta t}\,{}_tp_{40+k}^{01}\mu_{40+k+t}^{12}dt + 5{,}000\int_0^{5-k} e^{-\delta t}\,{}_tp_{40+k}^{01}dt - P\int_0^{5-k} e^{-\delta t}\,{}_tp_{40+k}^{00}dt
$$

$$
{}_kV^{(1)} = 10{,}000\int_0^{5-k} e^{-\delta t}\,{}_tp_{40+k}^{10}\mu_{40+k+t}^{02}dt + 15{,}000\int_0^{5-k} e^{-\delta t}\,{}_tp_{40+k}^{11}\mu_{40+k+t}^{12}dt + 5{,}000\int_0^{5-k} e^{-\delta t}\,{}_tp_{40+k}^{11}dt - P\int_0^{5-k} e^{-\delta t}\,{}_tp_{40+k}^{10}dt
$$

Homework 52.4.18

An insurer issues a 5-year sickness policy to a healthy life aged 40. You are given:

- $\delta = 0.06$
- The annual rate of premium, P, is payable continuously while the insured is healthy. Premium is determined under the equivalence principle.
- Death benefit: there's no death benefit.
- Sickness benefit: 5,000 payable continuously while the insured is sick



(a) If the death rate for the sick has fallen and the transition intensity from the sick to the healthy has also fallen, will P go up or down or stay the same?

(b) If the death rate for the sick has increased and the transition intensity from the healthy to the sick has also increased, will P go up or down or stay the same?

Homework Solution 52.4.18

★★★☆☆ Difficulty

(a) The insured will be sick longer, causing the sickness benefit and the premium P to go up.

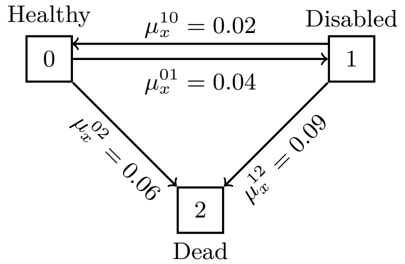
(b) It's not clear whether the premium will go or down.

Also see MLC Written Answer Sample Questions Q4, Q5, and Q12; Spring 2017 WA Q1.

52.5 Today's challenge

Homework 52.5.1

An insurance company uses the following continuous Markov model for pricing a combined 20-year disability, annuity, and life insurance contract.



The policy is issued to a healthy life age (x).

The transition intensities are constants for all ages.

$\delta = 0.05$

Calculate the EPV of each benefit separately.

Note. Not all benefits can be valued exactly. If the exact numerical solution might not exit, just write down the integral form for that EPV.

(A) 1000 per year payable continuously while the insured is healthy

(B) 1000 per year payable continuously while the insured is healthy but no payment is made if the insured is healthy after recovering from disability

(C) 1000 payable at the moment of death

(D) 1000 payable immediately when the insured is disabled for the 1st time

(E) 1000 payable immediately upon death or disability but no death benefit is paid if there's a prior disability claim

(F) 1000 per year payable continuously while the insured is disabled

(G) 1000 per year payable continuously throughout the 1st period of disability

(H) 1000 per year payable continuously throughout the 1st period of disability subject to a 6-month waiting period

(I) 1000 payable at the moment of death and an additional 500 if the insured was disabled at death

(J) a 1000 no-claim bonus payable at the end of the term if there's no death or disability claim during the term

(K) 1000 per year payable continuously while the insured is continuously disabled in excess of 6 months. However, any benefit period is limited to 5 years, but the number of the benefit periods can be unlimited.