Contents

1.1 ★★★★☆ Difficulty

Death benefit contracts where a sum is paid upon death are underwritten. Under such contracts higher mortality is bad for the insurer; the insurer uses underwriting to put insureds into different risk classes (such as smoker and non smoker, standard and substandard) so higher mortality risk classes pay higher premiums.

In contrast, if a contract pays some amounts while the annuitant is alive (such as pure endowment and annuity), higher mortality is good for the insurer and typically no underwriting is done. Only healthy people have an incentive to buy a pure endowment or an annuity contract. There's no need for an insurer to

Imagine that you are the death bed. You would love to buy a large life insurance policy but you would never want to buy an annuity contract.

The answer is C.

1.2 ★★★★☆ Difficulty

E is wrong. The opposite is true. For example, an ROP (return of premium) term policy and UL (universal life) have both the survival benefit and the death benefit. A deferred life annuity often pays a death benefit during the deferrable period.

2.1 ★★★★☆ Difficulty

$$e_{x:\overline{n}|}=\frac{\ell_{x+1}+\ell_{x+2}+\ldots+\ell_{x+n}}{\ell_x}$$

$$=\frac{\ell_0 S_0(x+1)+\ell_0 S_0(x+2)+\ldots+\ell_0 S_0(x+n)}{\ell_0 S_0(x)}=\frac{S_0(x+1)+S_0(x+2)+\ldots+S_0(x+n)}{S_0(x)}$$

$$e_x=\frac{\ell_{x+1}+\ell_{x+2}+\ldots+\ell_{x+\infty}}{\ell_x}=\frac{S_0(x+1)+S_0(x+2)+\ldots+S_0(x+\infty)}{S_0(x)}$$

$$\Rightarrow\quad e_{106}=\frac{S_0(107)+S_0(108)+\ldots+S_0(106+\infty)}{S_0(106)}$$

$$e_{106}=\frac{\left(1-\frac{107}{\omega}\right)^{1/4}+\left(1-\frac{108}{\omega}\right)^{1/4}+\ldots+\left(1-\frac{\omega-1}{\omega}\right)^{1/4}+\left(1-\frac{\omega}{\omega}\right)^{1/4}}{\left(1-\frac{106}{\omega}\right)^{1/4}}$$

$$\mu_x(t)=\mu(x+t)=-\frac{d}{dt}\ln {}_tp_x=-\frac{d}{dt}\ln\frac{S_0(x+t)}{S_0(x)}$$

$$=-\frac{d}{dt}\ln S_0(x+t)\text{ because }S_0(x)\text{ is a constant}$$

$$=-\frac{d}{dt}\ln\left(1-\frac{x+t}{\omega}\right)^{1/4}=\frac{1}{4}\times\frac{\frac{1}{\omega}}{1-\frac{x+t}{\omega}}$$

$$=\frac{1}{4}\times\frac{1}{\omega-(x+t)}$$

$$\mu(65)=\frac{1}{4}\times\frac{1}{\omega-65}=\frac{1}{180},\quad\omega=110$$

$$e_{106}=\frac{\left(1-\frac{107}{110}\right)^{1/4}+\left(1-\frac{108}{110}\right)^{1/4}+\left(1-\frac{109}{110}\right)^{1/4}+\left(1-\frac{110}{110}\right)^{1/4}}{\left(1-\frac{106}{110}\right)^{1/4}}=2.4786081$$

Answer: B

2.2 ★★★★★ Difficulty

Method 1. The number of survivors at $t=2$ is:

$$S=X_1+X_2+\ldots+X_{100{,}000}$$

where X_i is an indicator variable for the i-th person:

$$X_i=\begin{cases}1 & \text{If Person } i \text{ is alive at } t=2\\ 0 & \text{If Person } i \text{ is dead at } t=2\end{cases}$$

$$\begin{cases}X_i|\text{ have vaccine } \sim \text{bernoulli } p=(1-0.02)(1-0.01)=0.9702\\ X_i|\text{ no vaccine } \sim \text{bernoulli } p=((1-0.02)(1-0.02)=0.9604\end{cases}$$

- X_i|have vaccine's are *iid*

- $X_i|$not vaccine's are *iid*
- X_i's are correlated and not *iid* because the vaccine, if available, is given to all. For X_i's to be *iid*, no two people will have the same mortality shock

This is the most critical insight:

$$S = \begin{cases} \sim \text{ binomial } (n = 100000, p = 0.9702) \text{ prob: } 0.2 \\ \sim \text{ binomial } (n = 100000, p = 0.9604) \text{ prob: } 0.8 \end{cases}$$

$$\text{if } Y \sim \text{binomial } (n, p) \quad \Rightarrow \quad E(Y) = np, Var(Y) = npq, E\left(Y^2\right) = (np)^2 + npq$$

$$Var(S) = E\left(S^2\right) - E^2(S)$$

$$E(S) = P(\text{have vaccine})E(S|\text{ have vaccine}) + P(\text{no vaccine})E(S|\text{ no vaccine})$$
$$= 0.2 \times 100000 \times 0.9702 + 0.8 \times 100000 \times 0.9604 = 96,236$$

$$E(S^2) = P(\text{have vaccine})E(S^2|\text{ have vaccine}) + P(\text{no vaccine})E(S^2|\text{ no vaccine})$$
$$= 0.2\left(100000^2 \times 0.9702^2 + 100000 \times 0.9702(1 - 0.9702)\right) + 0.8\left(100000^2 \times 0.9604^2 + 100000 \times 0.9604(1 - 0.9604)\right)$$
$$= 9,261,524,980.79$$
$$Var(S) = 9,261,524,980.79 - 96,236^2 = 396.59^2$$

Method 2. This problem is modeled after AMLCR Example 11.4. If you copy AMLCR Example 11.4's solution using $N = 100,000$, we'll get the correct answer. This is because the variance of the deaths $Var[D(N)]$ is the same as the variance of the survivors $Var[S(N)]$.

First, replace AMLCR Example 11.4 q_{70} with the following:

$$q_{70} = \begin{cases} 1 - 0.9702 = 0.0298 & \text{with probability } 0.2 \\ 1 - 0.9604 = 0.0396 & \text{with probability } 0.8 \end{cases}$$

$$E[D] = N(0.2 \times 0.0298 + 0.8 \times 0.0396) = 0.03764N$$

$$V[E(D)] = 0.2(0.0298 - 0.03764)^2N^2 + 0.8(0.0396 - 0.03764)^2N^2 = 0.0000153664N^2$$

$$E[V(D)] = 0.2 \times 0.0298(1 - 0.0298)N + 0.8 \times 0.0396(1 - 0.0396)N = 0.036207864N$$

$$V[D] = 0.0000153664N^2 + 0.036207864N = 0.0000153664 \times 100000^2 + 0.036207864 \times 100000 = 396.59^2$$

Watch out the common trap of assuming that X_i's are *iid*:

$$S = X_1 + X_2 + \ldots + X_{100,000} \text{ OK}$$

$$X_1, X_2, \ldots, X_{100,000} \text{ are iid bernoulli random variables !!! Wrong !!!}$$
$$Var(S) = 100,000Var(X) \quad \text{!!! Wrong !!!}$$

Now let's modify the problem. Instead of everyone getting the vaccine, each person has 0.2 chance of getting the vaccine indepently of whether any other person gets the vaccine or not. Now X_i's are independent.

$$S \sim \text{binomial } (n = 100,000, p = 0.2(0.9702) + 0.8(0.9604) = 0.96236)$$

$$Var(S) = npq = 100000 \times 0.96236(1 - 0.96236) = 3622.323 = 60.185738^2$$

$Var(S)$ is much smaller because each life is independent now.

2.3 ★★★★★ Difficulty

Gompertz law of mortality: $\mu_x = Bc^x$, $0 < B < 1$, $c > 1$

$$f_{50}(10) = {}_{10}p_{50}\, \mu_{60}$$

$${}_{10}p_{50} = \exp\left(-\int_0^{10} \mu_{50+s} ds\right) = \exp\left(-\int_0^{10} 0.00027 \times 1.1^{50+s} ds\right)$$

$$\int_0^{10} 0.00027 \times 1.1^{50+s} ds = 0.00027 \times 1.1^{50} \int_0^{10} 1.1^s ds$$

$$x = e^{\ln x} \quad \Rightarrow \quad 1.1^s = e^{\ln 1.1^s} = e^{s \ln 1.1}$$

$$\int_0^{10} 1.1^s ds = \int_0^{10} e^{s \ln 1.1} ds = \frac{e^{10 \ln 1.1} - 1}{\ln 1.1} = 16.721639$$

$$_{10}p_{50} = \exp\left(-0.00027 \times 1.1^{50} \times 16.721639\right) = 0.58860425$$

$$f_{50}(10) = 0.58860425 \times 0.00027 \times 1.1^{60} = 0.048389$$

2.4 ★★★★☆ Difficulty

$$\mathring{e}_{75:\overline{10}|} = \int_0^{10} {}_tp_{75} dt$$

$$_{75+t}p_0 = {}_{75}p_0 \, {}_tp_{75}$$

$$\Rightarrow {}_tp_{75} = \frac{_{75+t}p_0}{_{75}p_0} = \frac{1 - \dfrac{(75+t)^2}{10000}}{1 - \dfrac{75^2}{10000}}$$

$$\int_0^{10} {}_tp_{75} dt = \frac{1}{1 - \dfrac{75^2}{10000}} \int_0^{10} \left(1 - \frac{(75+t)^2}{10000}\right) dt$$

Set $75 + t = y$.

$$\int_0^{10} \left(1 - \frac{(75+t)^2}{10000}\right) dt = \int_{75}^{85} \left(1 - \frac{y^2}{10000}\right) dy = 10 - \frac{1}{3} \times \frac{85^3 - 75^3}{10000} = 3.5916667$$

$$\mathring{e}_{75:\overline{10}|} = \frac{3.5916667}{1 - \dfrac{75^2}{10000}} = 8.2095239$$

2.5 ★★★★☆ Difficulty

$$i = 0 \quad \Rightarrow \quad e_{x:\overline{n}|} = a_{x:\overline{n}|}$$

Let $i = 0$. Then $v = 1$.

$$a_{40} = a_{40:\overline{20}|} + {}_{20}p_{40} v \, a_{60} = 18 + 0.8 \times 25 = 38$$

$$a_{40} = {}_1p_{40} v \left(1 + a_{41}\right)$$

$$38 = 0.997 \left(1 + a_{41}\right) \Rightarrow a_{41} = 37.114343$$

3.1 ★★★★☆ Difficulty

For a non-negative integer n and $0 \leq t \leq 1$, if the force of mortality between age $[x] + n$ and $[x] + n + 1$ is a constant $\mu_{[x]+n}$, then

$$\begin{cases} {}_1p_{[x]+n} = e^{-\mu_{[x]+n}} = \dfrac{\ell_{[x]+n+1}}{\ell_{[x]+n}} \\ {}_tp_{[x]+n} = e^{-t\mu_{[x]+n}} = \left(\dfrac{\ell_{[x]+n+1}}{\ell_{[x]+n}}\right)^t \\ \ell_{[x]+n+t} = \ell_{[x]+n} {}_tp_{[x]+n} = \ell_{[x]+n} \left(\dfrac{\ell_{[x]+n+1}}{\ell_{[x]+n}}\right)^t = \left(\ell_{[x]+n}\right)^{1-t} \left(\ell_{[x]+n+1}\right)^t \end{cases}$$

$$_{n|m}q_{[x]+k} = \frac{\ell_{[x]+k+n} - \ell_{[x]+k+n+m}}{\ell_{[x]+k}}$$

If the selection period is m years, then $k \geq m$,

$$[x] + k = x + k$$

$$\ell_{[x]+k} = \ell_{x+k}$$

$$_{2|3}q_{[60]+0.75} = \frac{\ell_{[60]+0.75+2} - \ell_{[60]+0.75+2+3}}{\ell_{[60]+0.75}} = \frac{\ell_{[60]+2.75} - \ell_{65.75}}{\ell_{[60]+0.75}}$$

$$= \frac{\left(\ell_{[60]+2}\right)^{0.25}\left(\ell_{[60]+3}\right)^{0.75} - (\ell_{65})^{0.25}(\ell_{66})^{0.75}}{\left(\ell_{[60]}\right)^{0.25}\left(\ell_{[60]+1}\right)^{0.75}}$$

$$= \frac{\left(\ell_{[60]+2}\right)^{0.25}(\ell_{603})^{0.75} - (\ell_{65})^{0.25}(\ell_{66})^{0.75}}{\left(\ell_{[60]}\right)^{0.25}\left(\ell_{[60]+1}\right)^{0.75}}$$

$$= \frac{7.7^{0.25} \times 7.4^{0.75} - 6.7^{0.25} \times 6.5^{0.75}}{8^{0.25} \times 7.9^{0.75}} = 0.11665193$$

The last step is based on the following insight:

In a mortality table, what matters is the ratio of the population, not the absolute size of the poluation. To simplify the calculation, we can scale down the population by 10,000:

x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
60	8.0	7.9	7.7	7.4	63
61	7.8	7.6	7.3	7.0	64
62	7.5	7.2	6.9	6.7	65
63	7.1	6.8	6.6	6.5	66

3.2 ★★★★☆ Difficulty

If deaths are uniformly distributed over integral ages (e.g. UDD), then the ${}_tp_{[x]}$ curve is a straight line connecting the following data points:

$${}_0p_{[x]}, {}_1p_{[x]}, {}_2p_{[x]}, \ldots$$

$\mathring{e}_{[x]:\overline{n}|} = \int_0^n {}_tp_x dt$ is always (UDD or not) the area under the curve ${}_tp_{[x]}$ bounded by $t=0$ and $t=n$. Under UDD, this area consists of n trapezoids, yielding:

$$\mathring{e}_{[x]:\overline{n}|} = \int_0^n {}_tp_x dt$$

$$= \frac{1}{2}\left({}_0p_{[x]} + {}_1p_{[x]}\right)$$

$$+\frac{1}{2}\left({}_1p_{[x]} + {}_2p_{[x]}\right)$$

$$+\ldots$$

$$+\frac{1}{2}\left({}_{n-1}p_{[x]} + {}_np_{[x]}\right)$$

$$= \frac{1}{2}\left({}_0p_{[x]} + {}_np_{[x]}\right) + {}_1p_{[x]} + {}_2p_{[x]} + \ldots + {}_{n-1}p_{[x]}$$

$$= \frac{1}{2}\left(1 + {}_np_{[x]}\right) + e_{[x]:\overline{n}|}$$

$$\begin{cases} \mathring{e}_{[x]:\overline{n}|} = \frac{1}{2}\left(1 + {}_np_{[x]}\right) + e_{[x]:\overline{n}|} \\ \mathring{e}_{[x]} = \frac{1}{2}(1+0) + e_{[x]} = 0.5 + e_{[x]} \end{cases}$$

$$\begin{cases} e_{[65]} = \dfrac{\ell_{[65]+1} + \ell_{[65]+2} + \ldots}{\ell_{[65]}} = \dfrac{\ell_{66} + \ell_{67} + \ldots}{\ell_{[65]}} = 15 - 0.5 = 14.5 \\ e_{[66]} = \dfrac{\ell_{[66]+1} + \ell_{[66]+2} + \ldots}{\ell_{[66]}} = \dfrac{\ell_{67} + \ell_{68} + \ldots}{\ell_{[66]}} \end{cases}$$

$$e_{[66]} = \frac{14.5\ell_{[65]} - \ell_{66}}{\ell_{[66]}} = \frac{14.5 \times 1000 - (1000 - 40)}{955} = 14.17801$$

$$\mathring{e}_{[66]} = 0.5 + 14.17801 = 14.67801$$

3.3 ★★★★☆ Difficulty

For a non-negative integer n and $0 \le t \le 1$, if deaths are uniformly distributed over age $[x]+n$ and $[x]+n+1$, then

$$\begin{cases} f_{[x]+n}(t) = q_{[x]+n} = 1 - \dfrac{\ell_{[x]+n+1}}{\ell_{[x]+n}} \\ {}_t q_{[x]+n} = \displaystyle\int_0^t f_{[x]+n}(s)ds = t\, q_{[x]+n} \\ \ell_{[x]+n+t} = \ell_{[x]+n}\, {}_t p_{[x]+n} = \ell_{[x]+n}\left(1 - t\left(1 - \dfrac{\ell_{[x]+n+1}}{\ell_{[x]+n}}\right)\right) = \ell_{[x]+n}(1-t) + \ell_{[x]+n+1} t \end{cases}$$

$$_{2.2}q_{[51]+0.5} = 1 - \frac{\ell_{[51]+2.7}}{\ell_{[51]+0.5}}$$

$$\ell_{[51]+0.5} = 0.5\left(\ell_{[51]} + \ell_{[51]+1}\right)$$

$$\ell_{[51]+2.7} = \ell_{53.7} = 0.3\ell_{53} + 0.7\ell_{54}$$

We scale down the population by 10,000.

$$_{2.2}q_{[51]+0.5} = 1 - \frac{\ell_{[51]+2.7}}{\ell_{[51]+0.5}} = 1 - \frac{0.3 \times 8.9 + 0.7 \times 8.3}{0.5(9.7 + 9.3)} = 0.10736842$$

3.4 ★★★★☆ Difficulty

Let S represent the number of survivors at age 95. We just need to find the 10%-th percentile of S, $Q_{10\%}(S)$. Since $P(S \le Q_{10\%}) = 0.1$, then it follows that $P(S > Q_{10\%}) = 0.9$. If we set N to be the interger of $Q_{10\%}$, since $N < Q_{10\%}$, then it follows that

$$P(S > N) > P(S > Q_{10\%}) = 0.9$$

$$S \sim \text{binomial}\left(n = 4,000, p = {}_{95-25}p_{25} = \frac{\ell_{95}}{\ell_{25}} = \frac{21,178.3}{99,871.1} = 0.212056\right)$$

$$S \approx\sim \text{normal}\left(\mu = np = 4000 \times 0.212056 = 848.23, \sigma = \sqrt{npq} = \sqrt{4000 \times 0.212056 \times 0.787944} = 25.85\right)$$

$$Q_{10\%}(S) = 848.23 + \Phi^{-1}(0.1)25.85 = 848.23 + (-1.2816)25.85 = 815.1$$

$$N = 815$$

Note that SOA used the normal approximation correction factor but I didn't. If there were another answer choice that is vey close to 815, then the correction factor is critical. Otherwise, the correction factor can be ignored.

3.5 ★★★★☆ Difficulty

First, let's scale down the population:

x	ℓ_x
60	9
61	8
62	7
63	6
64	5
65	4
66	3
67	2

$$_{3.4|2.5}q_{60} = \frac{\ell_{60+3.4} - \ell_{60+3.4+2.5}}{\ell_{60}} = \frac{\ell_{63.4} - \ell_{65.9}}{\ell_{60}}$$

$$\text{UDD } {}_{3.4|2.5}q_{60} = \frac{0.6\ell_{63} + 0.4\ell_{64} - (0.1\ell_{65} + 0.9\ell_{66})}{\ell_{60}} = \frac{0.6 \times 6 + 0.4 \times 5 - (0.1 \times 4 + 0.9 \times 3)}{9} = 0.27777778$$

$$\text{CF } {}_{3.4|2.5}q_{60} = \frac{(\ell_{63})^{0.6}(\ell_{64})^{0.4} - (\ell_{65})^{0.1}(\ell_{66})^{0.9}}{\ell_{60}} = \frac{6^{0.6} \times 5^{0.4} - 4^{0.1} \times 3^{0.9}}{9} = 0.27671612$$

$$100000(0.27777778 - 0.27671612) = 106.166$$

3.6 ★★★★☆ Difficulty

$$e_{64} = \frac{\ell_{65} + \ell_{66} + \ldots}{\ell_{64}} \quad \Rightarrow \ell_{65} + \ell_{66} + \ldots = \ell_{64} e_{64}$$

$$e_{[61]} = \frac{\ell_{[61]+1} + \ell_{[61]+2} + \ell_{[61]+3} + \ell_{[61]+4} + \ldots}{\ell_{[61]}}$$

$$= \frac{\ell_{[61]+1} + \ell_{[61]+2} + \ell_{64} + \ell_{65} + \dots}{\ell_{[61]}}$$

$$= \frac{\ell_{[61]+1} + \ell_{[61]+2} + \ell_{64}\left(1 + e_{64}\right)}{\ell_{[61]}}$$

$$\text{Set } \ell_{[61]} = 1$$

$$\ell_{[61]+1} = 1 - 0.1 = 0.9$$

$$\ell_{[61]+2} = 0.9(1 - 0.12)$$

$$\ell_{[61]+3} = \ell_{64} = 0.9(1 - 0.12)(1 - 0.14)$$

$$e_{[61]} = \frac{\ell_{[61]+1} + \ell_{[61]+2} + \ell_{64}\left(1 + e_{64}\right)}{\ell_{[61]}} = \frac{0.9 + 0.9(1 - 0.12) + 0.9(1 - 0.12)(1 - 0.14)(1 + 5.12)}{1} = 5.846832$$

3.7 ★★★★☆ Difficulty

$$_{2.5}q_{[50]+0.4} = 1 - \frac{\ell_{[50]+0.4+2.5}}{\ell_{[50]+0.4}} = 1 - \frac{\ell_{52.9}}{\ell_{[50]+0.4}}$$

- Set $\ell_{[50]} = 1$.
- $\ell_{[50]+1} = 1 - 0.005 = 0.995$
- $\ell_{52} = 0.995(1 - 0.0063) = 0.9887315$
- $\ell_{53} = 0.995(1 - 0.0063)(1 - 0.008) = 0.98082165$

$$_{2.5}q_{[50]+0.4} = 1 - \frac{\ell_{52.9}}{\ell_{[50]+0.4}} = 1 - \frac{0.9887315^{0.1} \times 0.98082165^{0.9}}{1^{0.6} \times 0.995^{0.4}} = 0.01642$$

3.8 ★★★★☆ Difficulty

This is similar to 3.4.

- Let S_1 represent the number of survivors at $t = 40$ out of the 1000 people age 35 today
- Let S_2 represent the number of survivors at $t = 40$ out of the 1000 people age 45 today

$$S_1 \sim \text{binomial}\left(n = 1000, p = \frac{\ell_{75}}{\ell_{35}} = \frac{85,203.5}{99,556.7} = 0.855829\right)$$

$$S_2 \sim \text{binomial}\left(n = 1000, p = \frac{\ell_{85}}{\ell_{45}} = \frac{61,184.9}{99,033.9} = 0.617818\right)$$

$$S_1 \approx\sim \text{normal}\ (\mu = 1000 \times 0.855829 = 855.83, Var = 1000 \times 0.855829(1 - 0.855829) = 123.39)$$

$$S_2 \approx\sim \text{normal}\ (\mu = 1000 \times 0.617818 = 617.82, Var = 1000 \times 0.617818(1 - 0.617818) = 236.12)$$

$$S = S_1 + S_2$$

$$S \approx\sim \text{normal}\ (\mu = E(S_1) + E(S_2) = 855.83 + 617.82 = 1473.65, Var = Var(S_1) + Var(S_2) = 123.39 + 236.12 = 18.96^2)$$

95%-th percentile of S is:

$$Q_{95\%}(S) = 1473.65 + \Phi^{-1}(0.95)18.96 = 1473.65 + (1.645)18.96 = 1504$$

3.9 ★★★★☆ Difficulty

This is similar to 3.4 and 3.8. Know one and you'll know them all.

- Let S_1 represent the number of survivors at $t = 25$ out of the 2000 people age 20 today
- Let S_2 represent the number of survivors at $t = 25$ out of the 2000 people age 45 today

$$S_1 \sim \text{binomial}\left(n = 2000, p = \frac{\ell_{45}}{\ell_{20}} = \frac{99,033.9}{100,000.0} = 0.990339\right)$$

$$S_2 \sim \text{binomial}\left(n = 2000, p = \frac{\ell_{70}}{\ell_{45}} = \frac{91,082.4}{99,033.9} = 0.919709\right)$$

$$S_1 \approx\sim \text{normal}\ (\mu = 2000 \times 0.990339 = 1,980.68, Var = 2000 \times 0.990339(1 - 0.990339) = 19.14)$$

$$S_2 \approx\sim \text{normal}\ (\mu = 2000 \times 0.919709 = 1,839.42, Var = 2000 \times 0.919709(1 - 0.919709) = 147.69)$$

$$S = S_1 + S_2$$

$$S \approx\sim \text{normal}\ (\mu = E(S_1) + E(S_2) = 1,980.68 + 1,839.42 = 3,820.10, Var = Var(S_1) + Var(S_2) = 19.14 + 147.69 = 12.92^2)$$

99%-th percentile of S is:

$$Q_{99\%}(S) = 3820.10 + \Phi^{-1}(0.99)12.92 = 3820.10 + (2.326)12.92 = 3850.15$$

3.10 ★★★★☆ Difficulty

t	ℓ_t^L	ℓ_t^R
0	75	25
1	$75 \times 0.75 + 35$	$25 \times 0.5 + 15$
2	$75 \times 0.75^2 + 35 \times 0.75 + 35$	$25 \times 0.5^2 + 15 \times 0.5 + 15$
3	$75 \times 0.75^3 + 35 \times 0.75^2 + 35 \times 0.75 + 35$	$25 \times 0.5^3 + 15 \times 0.5^2 + 15 \times 0.5 + 15$
...	...	...

$$\ell_5^L = 75 \times 0.75^5 + 35\left(0.75^4 + 0.75^3 + 0.75^2 + 0.75\right) = 75 \times 0.75^5 + 35 \times \frac{0.75 - 0.75^5}{1 - 0.75} = 89.575195$$

$$\ell_5^R = 25 \times 0.5^5 + 15\left(0.5^4 + 0.5^3 + 0.5^2 + 0.5\right) = 25 \times 0.5^5 + 15 \times \frac{0.5 - 0.5^5}{1 - 0.5} = 14.84375$$

$$\frac{89.575195}{89.575195 + 14.84375} = 0.85784$$

4.1 ★★★★☆ Difficulty

$$E(Z) = 2A_{40} - {}_{20}E_{40}\, A_{60} = 2 \times 0.36987 - 0.51276 \times 0.6257 = 0.41891$$
$$Var(Z) = E\left(Z^2\right) - E^2(Z) = 0.24954 - 0.41891^2 = 0.27213^2$$

4.2 ★★★★☆ Difficulty

We have to use the brute force approach to test at what death time the condition $Z > 2.77$ holds. To simplify calculation, set 100,000 death benefit as 1.

start	end	if	DB	$disF = 1.09^{-2\times \text{end}}$	$Z = DB \times DisF$	target Z	$Z >$ target Z?
0.00	0.50	$0 < T_x \leq 0.5$	3.00	0.917431	2.75229	2.77	0
0.50	1.00	$0.5 < T_x \leq 1$	3.30	0.841680	2.77754	2.77	1
1.00	1.50	$1 < T_x \leq 1.5$	3.60	0.772183	2.77986	2.77	1
1.50	2.00	$1.5 < T_x \leq 2$	3.90	0.708425	2.76286	2.77	0

From the above table, we see

$$P(Z > 2.77) = P\left(0.5 < T_x \leq 1.5\right)$$

$$\text{set } \ell_x = 1$$
$$\ell_{x+1} = 1 - 0.16 = 0.84$$
$$\ell_{x+2} = (1 - 0.16)(1 - 0.23) = 0.6468$$

$$\ell_{x+0.5} = \ell_x^{0.5} \times \ell_{x+1}^{0.5} = 1^{0.5} \times 0.84^{0.5}$$
$$\ell_{x+1.5} = \ell_{x+1}^{0.5} \times \ell_{x+2}^{0.5} = 0.84^{0.5} \times 0.6468^{0.5}$$

$$P\left(0.5 < T_x \leq 1.5\right) = \frac{\ell_{x+0.5} - \ell_{x+1.5}}{\ell_x} = \frac{1^{0.5} \times 0.84^{0.5} - 0.84^{0.5} \times 0.6468^{0.5}}{1} = 0.179418$$

BTW, since T_x is continuous, $P\left(0.5 < T_x \leq 1.5\right) = P\left(0.5 \leq T_x \leq 1.5\right)$.

4.3 ★★★★☆ Difficulty

$$A_{60:\overline{3}|} = q_{60} v + {}_{1|1}q_{60}\, v^2 + \left(1 - q_{60} - {}_{1|1}q_{60}\right) v^2$$

$$\begin{cases} A_{60:\overline{3}|\,|i=0.050} = 0.01 \times 1.050^{-1} + {}_{1|1}q_{60} \times 1.050^{-2} + \left(1 - 0.01 - {}_{1|1}q_{60}\right) 1.050^{-3} = 0.86545 \\ A_{60:\overline{3}|\,|i=0.045} = 0.01 \times 1.045^{-1} + {}_{1|1}q_{60} \times 1.045^{-2} + \left(1 - 0.01 - {}_{1|1}q_{60}\right) 1.045^{-3} \end{cases}$$

$$\begin{cases} {}_{1|1}q_{60} = \dfrac{0.86545 - 0.01 \times 1.050^{-1} - (1 - 0.01)1.050^{-3}}{1.050^{-2} - 1.050^{-3}} = 0.01683 \\ A_{60:\overline{3}|\,|i=0.045} = 0.01 \times 1.045^{-1} + 0.016831 \times 1.045^{-2} + (1 - 0.01 - 0.016831)1.045^{-3} = 0.8778 \end{cases}$$

4.4 ★★★★☆ Difficulty

$$E(Z) = \int_0^\infty b_t v(t)\, {}_tp_{40} \mu_{40+t} dt = \int_0^{40} (1 + 0.2t)(1 + 0.2t)^{-2} 0.025 dt$$
$$= \frac{0.025}{0.2}\left[\ln(1 + 0.2t)\right]_0^{40} = \frac{0.025}{0.2} \times \left(\ln(1 + 0.2 \times 40) - \ln(1 + 0.2 \times 0)\right) = 0.27465307$$

$$E(Z^2) = \int_0^\infty \left(b_t v(t)\right)^2 {}_tp_{40} \mu_{40+t} dt = \int_0^{40} (1 + 0.2t)^{-2} 0.025 dt$$
$$= \frac{0.025}{-0.2}\left[(1 + 0.2t)^{-1}\right]_0^{40} = \frac{0.025}{-0.2} \times \left((1 + 0.2 \times 40)^{-1} - (1 + 0.2 \times 0)^{-1}\right) = 0.11111111$$

$$Var(Z) = 0.11111111 - 0.27465307^2 = 0.035676801$$

When calculating $E(Z^2)$, never square the probability term such $f_x(t) = {}_tp_x\mu_{x+t}$ and q_x. Always square the payment b_t and the discounting factor $v(t)$.

4.5 ★★★★☆ Difficulty

Set 1000 death benefit as 1.

$$\begin{cases} E(Z) = vq_{45}^{(1)} + Fvq_{45}^{(2)} = v\,(0.04 + 0.2F) \\ E(Z^2) = v^2q_{45}^{(1)} + F^2v^2q_{45}^{(2)} = v^2\left(0.04 + 0.2F^2\right) \\ Var(Z) = v^2\left(0.04 + 0.2F^2\right) - v^2\,(0.04 + 0.2F)^2 \end{cases}$$

$$\frac{d}{dF}Var(Z) = v^2 \times 0.2 \times 2F - v^2 \times 2 \times 0.2(0.04 + 0.2F) = 0$$

$$F - (0.04 + 0.2F) = 0, \quad F = 0.05$$

$$\frac{d^2}{dF^2}Var(Z) = v^2 \times 0.2 \times 2 - v^2 \times 2 \times 0.2(0.2) > 0 \quad \Rightarrow \quad Var(Z) \text{ reaches its min at } F = 0.05$$

$$1000F = 50$$

4.6 ★★★★☆ Difficulty

q_x^{SULT} can be found from the Std Ult Life Table.

x	q_x^{SULT}	q_x loading	q_x
70	0.010413	1	0.010413
71	0.011670	0.95	0.011087
72	0.013081	0.9025	0.011805

$$A_{\overset{1}{70}:\overline{3}|} = vq_{70} + v^2\,{}_1p_{70}q_{71} + v^3\,{}_2p_{70}q_{72}$$

$$= 1.05^{-1}(0.010413) + 1.05^{-2}(1 - 0.010413)0.011087 + 1.05^{-3}(1 - 0.010413)(1 - 0.011087)0.011805 = 0.029848$$

$$1000 \times 0.029848 = 29.848$$

4.7 ★★★★☆ Difficulty

$$\begin{cases} 72 = d_{97} = \ell_{97}q_{97} = \ell_{97} \\ \ell_{97} = \ell_{96}p_{96} = \ell_{96}0.2 \\ \ell_{96} = \ell_{95}p_{95} = \ell_{95}0.6 \end{cases}$$

$$\begin{cases} \ell_{97} = 72 \\ \ell_{96} = 72/0.2 = 360 \\ \ell_{95} = \ell_{96}/0.6 = 600 \end{cases}$$

$$\frac{\ell_{93} - \ell_{95.5}}{\ell_{90}} = \frac{825 - 0.5(600 + 360)}{1000} = 0.345$$

4.8 ★★★★☆ Difficulty

$$\begin{aligned} A_{50} &= 1.05^{-1}q_{50} + 1.05^{-2}\,{}_{1|1}q_{50} + 1.05^{-3}\,{}_{2|1}q_{50} + \ldots = 0.18931 \\ A^*_{50} &= 1.04^{-1}q_{50} + 1.04^{-1}1.05^{-1}\,{}_{1|1}q_{50} + 1.04^{-1}1.05^{-2}\,{}_{2|1}q_{50} + \ldots \\ &= 1.04^{-1}q_{50} + \frac{1.04^{-1}}{1.05^{-1}}\left(1.05^{-2}\,{}_{1|1}q_{50} + 1.05^{-3}\,{}_{2|1}q_{50} + \ldots\right) \\ &= 1.04^{-1}q_{50} + \frac{1.04^{-1}}{1.05^{-1}}\left(A_{50} - 1.05^{-1}q_{50}\right) \\ &= 1.04^{-1} \times 0.001209 + \frac{1.04^{-1}}{1.05^{-1}}\left(0.18931 - 1.05^{-1} \times 0.001209\right) = 0.19113 \end{aligned}$$

4.9 ★★★☆☆ Difficulty

$$\begin{aligned} A_{35} &= A_{\overset{1}{35}:\overline{15}|} + {}_{15}E_{35}A_{50} \\ &= A_{\overset{1}{35}:\overline{15}|} + \left(A_{35:\overline{15}|} - A_{\overset{1}{35}:\overline{15}|}\right)A_{50} \end{aligned}$$

$$0.32 = 0.25 + (0.39 - 0.25)A_{50}, \quad A_{50} = 0.5$$

4.10 ★★★★☆ Difficulty

There's no shortcut other than checking each answer choice.

- A is wrong because it implies that the death benefit is 1 if $30 < T_x < \infty$. A would have been correct if it were written as ${}_{10|}\overline{A}_x + {}_{20|}\overline{A}_x - 2\,{}_{30|}\overline{A}_x$.

- B is wrong because its first term implies that the death benefit is 1 if $0 < T_x < 10$. A would have been correct if its first term were written as ${}_{10}E_x\overline{A}_{x+10}$ instead of $\overline{A}_x$.
- C would have been correct if the first term were written as ${}_{10}E_x\overline{A}_{x+10}$.
- D is correct.
- E simply doesn't make any sense.

4.11 ★★★★☆ Difficulty

To simplify calculation, set 1000 benefit as 1. Then $E(Z_1) = 0.528$ and $Var(Z_2) = 15000/1000^2 = 0.015$.

$$E(Z_2) = A_{\stackrel{1}{x}:\overline{n}|} + A_{x:\stackrel{1}{\overline{n}|}} = 0.528 + 0.209 = 0.737$$
$$E(Z_2^2) = {}^2A_{\stackrel{1}{x}:\overline{n}|} + {}^2A_{x:\stackrel{1}{\overline{n}|}} = {}^2A_{\stackrel{1}{x}:\overline{n}|} + 0.136$$
$$Var(Z_2) = \left({}^2A_{\stackrel{1}{x}:\overline{n}|} + 0.136\right) - 0.737^2 = 0.015, \quad {}^2A_{\stackrel{1}{x}:\overline{n}|} = 0.422169$$
$$Var(Z_1) = 0.422169 - 0.528^2 = 0.143385, \quad 0.143385 \times 1000^2 = 143,385$$

4.12 ★★★★☆ Difficulty

$$Z_3 = 2Z_1 + Z_2$$
$$Var(Z_3) = 4Var(Z_1) + Var(Z_2) + 4Cov(Z_1, Z_2)$$

$$Cov(Z_1, Z_2) = E(Z_1Z_2) - E(Z_1)E(Z_1) = 0 - 1.65 \times 10.75$$

$Z_1Z_2 = 0$ because at least one of Z_1 and Z_2 is zero. If death occurs during the first 20 years, then $Z_2 = 0$; if death occurs after the first 20 years, $Z_1 = 0$.

$$Var(Z_3) = 4Var(Z_1) + Var(Z_2) + 4Cov(Z_1, Z_2) = 4 \times 46.75 + 50.78 - 4 \times 1.65 \times 10.75 = 166.83$$

